Applied Mathematics E-Notes, 20(2020), 199-203 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On The Structure Of Powers Of Toeplitz Matrices*

Mohamed Tahar Mezeddek†, Ismaiel Krim‡, Abderrahmane Smail§

Received 23 August 2019

Abstract

A finite Toeplitz matrix T is normal if and only if it is a rotation and a translation of a Hermitian Toeplitz matrix (Type **I**) or it is a generalised circulant (Type **II**). In this paper, we show that the powers of a Type II matrix T are also Type II.

1 Introduction

The purpose of the present paper is to study the normal structure of powers of finite normal Toeplitz matrices. There are already results related to the structure of finite or infinite normal Toeplitz matrices. An important result is given by Brown and Halmos in [1]. It is stated that an infinite Toeplitz matrix (operator) is normal if and only if it is a rotation and a translation of a Hermitian Toeplitz matrix. Ikramov [3] has also shown that a normal Toeplitz matrix (of order at most 4) over the real field must be symmetric, skew-symmetric, circulant, or skew-circulant. In [2] Farenick et al. state that every finite complex normal Toeplitz matrix T is a rotation and a translation of a Hermitian Toeplitz matrix (Type I), that is $T = \alpha I + \beta \mathcal{H}$, where α and β are complex numbers, and $\mathcal{H}$ is a Hermitian Toeplitz; or is a generalized circulant (Type II) Toeplitz matrix of the form

$$T = \begin{pmatrix} a_0 & a_N e^{i\theta} & \ddots & a_1 e^{i\theta} \\ a_1 & a_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_N e^{i\theta} \\ a_N & \ddots & a_1 & a_0 \end{pmatrix}$$

for some fixed real θ. Here we are interested in T^n, $n \in \mathbb{N}$, and try to find out whether it remains of the same type as T, be it Type I or II.

2 Type II Matrix

Let

$$T = \begin{pmatrix} a_0 & a_N e^{i\theta} & \ddots & a_1 e^{i\theta} \\ a_1 & a_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & a_N e^{i\theta} \\ a_N & \ddots & a_1 & a_0 \end{pmatrix} \tag{1}$$

be a normal Toeplitz matrix of Type II for some fixed real θ.

*Mathematics Subject Classifications: 20F05, 20F10, 20F55, 68Q42.

†University Oran1 Ahmed Ben Bella. Faculty of Exact and Applied Sciences. Department of Mathematics. B.P 1524 El M'naouer, 31100, Oran, Algeria. Laboratory of Geometry and Analysis - GENLAB, University Oran1, Algeria

‡Département de Mathématiques, Faculté des Sciences Exactes et Appliquées, Université Oran 1, Oran, Algérie

§Département de Mathématiques, Faculté des Sciences Exactes et Appliquées, Université Oran 1, Oran, Algérie

Theorem 1 *Each T^n, $n \in \mathbb{N}$, is of Type II.*

Proof. We proceed by induction. First we prove that T^2 is of Type II. Let T be the matrix given by (1). Then

$$T^2 = \begin{bmatrix} \alpha_{0\,0} & \alpha_{0\,1} & \ddots & \alpha_{0\,N} \\ \alpha_{1\,0} & \alpha_{1\,1} & \ddots & \ddots \\ \ddots & \ddots & \ddots & \alpha_{N-1\,N} \\ \alpha_{N\,0} & \ddots & \alpha_{N\,N-1} & \alpha_{N\,N} \end{bmatrix}. \tag{2}$$

To see this, we prove that:

$$\begin{cases} \alpha_{p\,p} = \alpha_{q\,q} & \forall\, p, q \in \{0, \cdots, N\}, \\ \alpha_{p\,q} = \alpha_{p+1\,q+1} & \forall\, p, q \in \{0, \cdots, N-1\}, \\ \alpha_{0\,N-p+1} = \alpha_{p\,0} e^{i\theta} & \forall\, p \in \{1, \cdots, N\}. \end{cases}$$

Step 1. We prove that $\alpha_{p\,p} = \alpha_{q\,q}$. Clearly, $\alpha_{p\,p}$ is given by

$$\begin{aligned} \alpha_{p\,p} \quad = \quad & (a_p a_{N-p+1} + a_{p-1} a_{N-p+2} + \cdots + a_2 a_{N-1} + a_1 a_N)\, e^{i\theta} + a_0^2 \\ & + (a_N a_1 + a_{N-1} a_2 + \cdots + a_{p+2} a_{N-p-1} + a_{p+1} a_{N-p})\, e^{i\theta}. \end{aligned}$$

We obtain

$$\alpha_{p\,p} = a_0^2 + \left(\sum_{k=1}^{N} a_k a_{N-k+1} \right) e^{i\theta} = \mathit{constant} \ \ \forall\, p \in \{0, \cdots, N\}.$$

Hence: $\alpha_{p\,p} = \alpha_{q\,q}$.

Step 2. We prove that $\alpha_{p\,q} = \alpha_{p+1\,q+1}$. We have to consider two cases:

Case 1: We assume that $p > q$. Then

$$\begin{aligned} \alpha_{p\,q} \quad = \quad & (a_p a_{N-q+1} + a_{p-1} a_{N-q+2} + \cdots + a_{p-q+1} a_N)\, e^{i\theta} \\ & + (a_{p-q} a_0 + a_{p-q-1} a_1 + \cdots + a_1 a_{p-q-1} + a_0 a_{p-q}) \\ & + (a_N a_{p-q+1} + a_{n-1} a_{p-q+2} + \cdots + a_{p+2} a_{N-q-1} + a_{p+1} a_{N-q})\, e^{i\theta}. \end{aligned}$$

We have

$$\alpha_{pq} = \sum_{k=0}^{p-q} a_{p-q-k} a_k + e^{i\theta} \left(\sum_{k=1}^{N-p+q} a_{p-q+k} a_{N-k+1} \right),$$

$$\begin{aligned} \alpha_{p+1\,q+1} \quad = \quad & (a_{p+1} a_{N-q} + a_p a_{N-q+1} + a_{p-1} a_{N-q+2} + \cdots + a_{p-q+1} a_N)\, e^{i\theta} \\ & + (a_{p-q} a_0 + a_{p-q+1} a_1 + \cdots + a_1 a_{p-q-1} + a_0 a_{p-q}) \\ & + (a_n a_{p-q+1} + a_{N-1} a_{p-q+2} + \cdots + a_{p+3} a_{N-q-2} + a_{p+2} a_{N-q-1})\, e^{i\theta} \\ = \quad & (a_p a_{N-q+1} + a_{p-1} a_{N-q+2} + \cdots + a_{p-q+1} a_n)\, e^{i\theta} \\ & + (a_{p-q} a_0 + a_{p-q-1} a_1 + \cdots + a_1 a_{p-q-1} + a_0 a_{p-q}) \\ & (a_N a_{p-q+1} + a_{N-1} a_{p-q+2} + \cdots + a_{p+2} a_{N-q-1} + a_{p+1} a_{N-q})\, e^{i\theta} \\ = \quad & a_{p\,q}. \end{aligned}$$

As a result, we see that $\alpha_{p\,q} = \alpha_{p+1\,q+1}$.

Case 2: We assume that $p < q$. Then

$$\begin{aligned}
\alpha_{p\,q} &= \left(a_p a_{N-q+1} + a_{p-1}a_{N-q+2} + \cdots + a_1 a_{N-q+p} + a_0 a_{N-q+p+1}\right) e^{i\theta} \\
&\quad + \left(a_N a_{N-q+p+2} + a_{N-1}a_{N-q+p+3} + \cdots + a_{N-q+p+3}a_{n-1} + a_{N-q+p+2}a_N\right) e^{2i\theta} \\
&\quad + \left(a_{N-q+p+1}a_0 + a_{N-q+p}a_N + a_{N-q+p-1}a_2 + \cdots + a_{p+2}a_{N-q-1} + a_{p+1}a_{N-q}\right) e^{i\theta} \\
&= e^{2i\theta}\left(\sum_{k=0}^{q-p-2} a_{N-k}a_{N-q+p+2+k}\right) + e^{i\theta}\left(\sum_{k=0}^{N-q+p+1} a_k a_{N-q+p+1-k}\right),
\end{aligned}$$

$$\begin{aligned}
\alpha_{p+1\,q+1} &= \left(a_{p+1}a_{N-q} + a_p a_{N-q+p+1} + a_{p-1}a_{N-q+2} + \cdots + a_1 a_{N-q+p} + a_0 a_{N-q+p+1}\right) e^{i\theta} \\
&\quad + \left(a_N a_{N-q+p+2} + a_{N-1}a_{N-q+p+3} + \cdots + a_{N-q+p+3}a_{N-1} + a_{N-q+p+2}a_N\right) e^{2i\theta} \\
&\quad + \left(a_{N-q+p+1}a_0 + a_{N-q+p}a_1 + a_{N-q+p-1}a_2 + \cdots + a_{p+3}a_{N-q-2} + a_{p+2}a_{N-q-1}\right) e^{i\theta} \\
&= \alpha_{p\,q}.
\end{aligned}$$

Step 3. We prove that $\alpha_{0\,N+1-p} = \alpha_{p\,0}e^{i\theta}$ where

$$\begin{aligned}
\alpha_{p\,0} &= \left(a_p a_0 + a_{p-1}a_1 + \cdots + a_1 a_{p-1} + a_0 a_p\right) \\
&\quad + \left(a_{p+1}a_N + a_{p+2}a_{N-1} + \cdots + a_{N-1}a_{p+2} + a_N a_{p+1}\right) e^{i\theta} \\
&= \sum_{k=0}^{p} a_{p-k}a_k + \left(\sum_{k=1}^{N-p} a_{p+k}a_{N-k+1}\right) e^{i\theta},
\end{aligned}$$

$$\begin{aligned}
\alpha_{0\,N-p+1} &= a_0 a_p e^{i\theta} + \left(a_N a_{p+1} + a_{N-1}a_{p+2} + \cdots + a_{p+2}a_{N-1} + a_{p+1}a_n\right) e^{2i\theta} \\
&\quad + \left(a_p a_0 + a_{p-1}a_1 + \cdots + a_2 a_{p-2} + a_1 a_{p-1}\right) e^{i\theta}.
\end{aligned}$$

We obtain $\alpha_{0\,N-p+1} = \alpha_{p\,0}e^{i\theta}$.

Thus, T^2 is of Type II. Assuming that the matrix T^n is of Type II,

$$T^n = \begin{pmatrix} t_0 & t_N e^{i\theta} & \ddots & t_1 e^{i\theta} \\ t_1 & t_0 & \ddots & \ddots \\ \ddots & \ddots\ddots & \ddots & t_N e^{i\theta} \\ t_N & \ddots & t_1 & t_0 \end{pmatrix}, \tag{3}$$

we prove that T^{n+1} is of the same type

$$T^{n+1} = T^n \times T = \begin{pmatrix} \beta_{0\,0} & \beta_{0\,1} & \ddots & \beta_{0\,N} \\ \beta_{1\,0} & \beta_{1\,1} & \ddots & \ddots \\ \ddots & \ddots & \ddots & \beta_{N-1\,N} \\ \beta_{N\,0} & \ddots & \beta_{N\,N-1} & \beta_{N\,N} \end{pmatrix}. \tag{4}$$

For this, it is sufficient to prove that

$$\begin{cases} \beta_{p\,p} = \beta_{q\,q} & \forall\, p,q \in \{0,\cdots,N\}, \\ \beta_{p\,q} = \beta_{p+1\,q+1} & \forall\, p,q \in \{0,\cdots,N-1\}, \\ \beta_{0\,N-p+1} = \beta_{p\,0}e^{i\theta} & \forall\, p \in \{1,\cdots,N\}. \end{cases} \tag{5}$$

In order to prove (5), we need the following three steps.

Step 1. We prove that $\beta_{p\,p} = \beta_{q\,q}$ where

$$\begin{aligned}\beta_{p\,p} &= t_0a_0 + (t_pa_{N-p+1} + t_{p-1}a_{N-P+2} + \cdots + t_1a_N)\,e^{i\theta} \\ &\quad + (t_Na_1 + t_{N-1}a_2 + \cdots + t_{p+1}a_{N-p})\,e^{i\theta} \\ &= t_0a_0 + \left(\sum_{k=1}^{N} t_ka_{N-k+1}\right)e^{i\theta} \qquad \forall p \in \{0, \cdots, N\}\,.\end{aligned}$$

Then we have: $\beta_{p\,p} = \beta_{q\,q}$.

Step 2. We prove that $\beta_{p\,q} = \beta_{p+1\,q+1}$. We have to consider two cases:

Case 1. We assume that $p > q$. Then

$$\begin{aligned}\beta_{p\,q} &= (t_pa_{N-q+1} + t_{p-1}a_{N-q+2} + \cdots + t_{p-q+1}a_N)\,e^{i\theta} + (t_{p-q}a_0 + t_{p-q-1}a_1 + \cdots + t_1a_{p-q-1} + t_0a_{p-q}) \\ &\quad + (t_Na_{p-q+1} + t_{N-1}a_{p-q+2} + \cdots + t_{p+2}a_{N-q-1} + t_{p+1}a_{N-q})\,e^{i\theta},\end{aligned}$$

$$\begin{aligned}\beta_{p+1\,q+1} &= (t_{p+1}a_{N-q} + t_pa_{N-q+1} + t_{p-1}a_{N-q+2} + \cdots + t_{p-q+1}a_N)\,e^{i\theta} \\ &\quad + (t_{p-q}a_0 + t_{p-q-1}a_1 + \cdots + t_1a_{p-q-1} + t_0a_{p-q}) \\ &\quad + (t_Na_{p-q+1} + t_{N-1}a_{p-q+2} + \cdots + t_{p+3}a_{N-q-2} + t_{p+2}a_{N-q-1})\,e^{i\theta} \\ &= \beta_{p\,q}.\end{aligned}$$

Case 2. We assume that $p < q$. Then

$$\begin{aligned}\beta_{p\,q} &= (t_pa_{N-q+1} + t_{p-1}a_{N-q+2} + \cdots + t_1a_{N-q+p} + t_0a_{N-q+p+1})\,e^{i\theta} \\ &\quad + (t_Na_{N-q+p+2} + t_{N-1}a_{N-q+p+3} + \cdots + t_{N-q+p+2}a_N)\,e^{2i\theta} \\ &\quad + (t_{N-q+p+1}a_0 + t_{N-q+p}a_1 + \cdots + t_{p+2}a_{N-q-1} + t_{p+1}a_{N-q})\,e^{i\theta},\end{aligned}$$

$$\begin{aligned}\beta_{p+1\,q+1} &= t_{p+1}a_{N-q}e^{i\theta} + (t_pa_{N-q+1} + t_{p-1}a_{N-q+2} + \cdots + t_1a_{N-q+p} + t_0a_{N-q+p+1})\,e^{i\theta} \\ &\quad + (t_Na_{N-q+p+2} + \cdots + t_{N-q+p+2}a_N)\,e^{2i\theta} \\ &\quad + (t_{N-q+p+1}a_0 + t_{N-q+p}a_1 + \cdots + t_{p+3}a_{N-q-2} + t_{p+2}a_{N-q-1})\,e^{i\theta} \\ &= \beta_{p\,q}.\end{aligned}$$

Step 3. We prove that $\beta_{0\,N-p+1} = \beta_{p\,0}e^{i\theta}$ where

$$\begin{aligned}\beta_{p\,0} &= (t_pa_0 + t_{p-1}a_1 + \cdots + t_1a_{p-1} + t_0a_p) \\ &\quad + (t_Na_{p+1} + t_{N-1}a_{p+2} + \cdots + t_{p+2}a_{N-1} + t_{p+1}a_N)\,e^{i\theta},\end{aligned}$$

$$\begin{aligned}\beta_{0\,N+1-p} &= t_0a_pe^{i\theta} + (t_Na_{p+1} + t_{N-1}a_{p+2} + \cdots + t_{p+2}a_{N-1} + t_{p+1}a_N)\,e^{2i\theta} \\ &\quad + (t_pa_0 + t_{p-1}a_1 + \cdots + t_2a_{p-2} + t_1a_{p-1})\,e^{i\theta} \\ &= \beta_{p\,0}e^{i\theta}.\end{aligned}$$

Consequently, we have obtained : T^n is of Type II $\forall n \in \mathbb{N}$. ■

Example 1 *Let*

$$A = \begin{pmatrix} 1 & 3e^{i\theta} & 2e^{i\theta} \\ 2 & 1 & 3e^{i\theta} \\ 3 & 2 & 1 \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

A simple calculation yields;

$$A^2 = \begin{pmatrix} 1+12e^{i\theta} & 10e^{i\theta} & (4+9e^{i\theta})e^{i\theta} \\ 4+9e^{i\theta} & 1+12e^{i\theta} & 10e^{i\theta} \\ 10 & 4+9e^{i\theta} & 1+12e^{i\theta} \end{pmatrix}$$

and

$$A^3 = \begin{pmatrix} 1+44e^{i\theta}+27e^{2i\theta} & (21+54e^{i\theta})e^{i\theta} & (6+63e^{i\theta})e^{i\theta} \\ 6+63e^{i\theta} & 1+44e^{i\theta}+27e^{2i\theta} & (21+54e^{i\theta})e^{i\theta} \\ 21+54e^{i\theta} & 6+63e^{i\theta} & 1+44e^{i\theta}+27e^{2i\theta} \end{pmatrix}.$$

3 Type I Matrix

In general, the n-th power of a normal Toeplitz matrix T of Type I is not always of type Type I. To see a counterexample, let $A = \alpha I + \beta \mathcal{H}$ be a normal Toeplitz matrix of type (**I**) with $\alpha = -2i$ and $\beta = 1$ and $\mathcal{H}$ be a Hermitian Toeplitz matrix given by

$$\mathcal{H} = \begin{pmatrix} 3+2i & 2-i & -1+3i \\ 2+i & 3+2i & 2-i \\ -1-3i & 2+i & 3+2i \end{pmatrix}.$$

Then we have

$$A = \begin{pmatrix} 3 & 2-i & -1+3i \\ 2+i & 3 & 2-i \\ -1-3i & 2+i & 3 \end{pmatrix}.$$

While a simple calculation gives us

$$A^2 = \begin{pmatrix} 24 & 7-i & -3+14i \\ 7+i & 19 & 7-i \\ -3-14i & 7+i & 24 \end{pmatrix}.$$

It is clear that A^2 is not of Type I merely because it is not a Toeplitz matrix.

References

[1] A. Brown and P. R. Halmos, Algebraic properties of Toeplitz operators., J. Reine Angew. Math., 213(1963/1964), 89–102.

[2] D. R. Farenick, M. Krupnik, N. Krupnik and W. Y. Lee, Normal Toeplitz Matrices., SIAM J. Matrix Anal. Appl., 17(1996), 1037–1043.

[3] Kh. D. Ikramov, “Classification of normal Toeplitz matrices with real entries”., Mat. Zametki, 57(1995), 670–680; Math. Notes, 57(1995), 463–469.

[4] Kh. D. Ikramov, Describing normal Toeplitz matrices., Zh. Vychisl. Mat. Mat. Fiz., 34(1994), 473–479.