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A Nonsymmetric Nash-Riccati Equation And Decoupled Schemes For A Stabilizing Solution*

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Abstract

We investigate a nonsymmetric Nash-Riccati equation which has arisen in linear quadratic games for positive systems. There are papers where the stabilising solution of the nonsymmetric Nash-Riccati equation is computed applying the Newton procedure in the literature. We construct a new decoupled iteration scheme for computing the stabilizing nonnegative solution of the nonsymmetric Nash-Riccati equation. The convergence properties of the proposed decoupled iteration are investigated and a sufficient condition for convergence is derived. The performance of the proposed algorithm is illustrated on some numerical examples. On the basis of the experiments we derive conclusions for applicability of the new scheme.

1 Introduction

We investigate the nonsymmetric matrix Riccati equation in the special form:

$$\mathfrak{R}(\mathcal{X}) = -\mathcal{D}\mathcal{X} - \mathcal{X}A + \mathcal{X}\mathcal{S}\mathcal{X} - \mathcal{Q} \,=\, 0\,. \tag{1}$$

The unknown matrix $\mathcal{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is not square. For this reason the matrix coefficients have different dimensions, i.e. they are : $(-A)$ is an $n\times n$ M-matrix, $\mathcal{D} = \begin{pmatrix} A^T & 0 \\ 0 & A^T \end{pmatrix}$, $\mathcal{S} = (S_1 \quad S_2)$ where S_i is an $n\times n$ nonpositive matrix, $i = 1,2$, $\mathcal{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ and Q_i is an $n\times n$ symmetric nonnegative matrix.

The linear quadratic differential game is described by the dynamic system:

$$\dot{x} = Ax + B_1u_1 + B_2u_2\,, \quad x(0) = x_0\,, \tag{2}$$

with matrices $A \in \mathbf{R}^{n\times n}$, $B_1 \in \mathbf{R}^{m_1}$, $B_2 \in \mathbf{R}^{m_2}$, $x(t) \in \mathbf{R}^n$ is the state of the game and the control functions u_1, u_2.

We say that a system is positive, if for nonnegative inputs u_1, u_2 and nonnegative initial values x_0, the state function x is nonnegative. A sufficient condition for the above system to be a positive system is that $(-A)$ is an M-matrix and B_1, B_2 are nonnegative matrices.

The cost-functional for each player is considered

$$J_i(u_1,u_2) = \int_0^\infty (x^T Q_i\, x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2)\, dt\,, \quad i = 1,2\,,$$

where $Q_i \in \mathbf{R}^{n\times n}$, $R_{ij} \in \mathbf{R}^{m_i\times m_j}$, $i,j = 1,2$. The matrices Q_i, R_{ij}, i, $j = 1,2$ are symmetric.

We say that (u_1^*, u_2^*) is an open loop Nash equilibrium if for each player $i = 1,2$, the inequalities

$$J_1(u_1^*,u_2^*) \geq J_1(u_1,u_2^*) \ \text{ and } \ J_2(u_1^*,u_2^*) \geq J_2(u_1^*,u_2)$$

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hold [12, 1]. According to the above inequalities the aim of each player is to maximize his own utility function. The notation an 'open loop' strategy means that the players have to choose their strategies u_1 and u_2 prior to the game and that their only information on the state of the game is the initial state x_0. A sufficient condition to exits for the existence of unique Nash equilibrium is the matrices R_{11}, R_{22} are negative definite ones (see [12, Theorem 4]). The optimal strategy of each player $u_i^*, i = 1, 2$ is given by $u_i^* = -R_{ii}^{-1} B_i^T \tilde{X}_i x^*$, where $\begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$ is the stabilizing solution of (1) with $S_i = B_i R_{ii}^{-1} B_i^T$ is an $n \times n$ nonpositive matrix, $i = 1, 2$, and x^* being the solution of the closed loop equation $\dot{x} = (A - S_1\tilde{X}_1 - S_2\tilde{X}_2)x, \ \ x(0) = x_0$ (see [12, Theorem 4]).

The stabilizing solution of (1) is defined in [12] as follows. A left-right stabilizing solution $\begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$ of (1) satisfies that the matrices $A - S_1\tilde{X}_1 - S_2\tilde{X}_2$ and $\begin{pmatrix} A^T - \tilde{X}_1 S_1 & -\tilde{X}_1 S_2 \\ -S_1\tilde{X}_2 & A^T - \tilde{X}_2 S_2 \end{pmatrix}$ are both stable. The Newton method under some conditions for computing the stabilizing solution of (1) is proposed in [12]. Authors in [1] present a study of the Nash equilibria on positive systems, described by the concept of deterministic feedback Nash equilibrium and the concept of open loop Nash equilibrium. We establish [10] a new decoupled recursive equations for computing the stabilizing solution of (1) called the Alternately Linearized Implicit Decoupled Iteration (ALIDI).

In this paper, we propose a New Linearized Implicit Decoupled Iteration (NLIDI) for computing the stabilizing nonnegative solution of (1). Compared to the ALIDI method, the new iteration is more efficient because the algorithm needs less matrix computations in the iterative process. Thus, it requires less CPU time for solving (1) than ALIDI and it is easy to construct a parallel version of NLIDI. Some numerical experiments are provided to confirm the effectiveness of the NLIDI.

The rest of the paper is organized as follows. In section 2, we describe the NLIDI and derive its convergence properties. In Section 3, we present numerical examples to illustrate the performance of the algorithm. Concluding remarks are given in Section 4.

The notation $\mathbf{R}^{s\times q}$ stands for $s \times q$ real matrices. In this investigation we exploit the properties of nonnegative matrices. A matrix $A = (a_{ij}) \in \mathbf{R}^{m\times n}$ is a nonnegative matrix if the inequalities $a_{ij} \geq 0$ are satisfied for all $1 \leq i \leq m$ and $1 \leq j \leq n$. We use an elementwise order relation. The inequality $P \geq Q (P > Q)$ for $P = (p_{ij})$, $Q = (q_{ij})$ means that $p_{ij} \geq q_{ij} (p_{ij} > q_{ij})$ for all indexes i and j. A matrix $A = (a_{ij}) \in \mathbf{R}^{n\times n}$ is said to be a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be written in the form $A = \alpha I - N$ with N being a nonnegative matrix. Each M-matrix is a Z-matrix with if $\alpha \geq \rho(N)$, where $\rho(N)$ is the spectral radius of N. It is called a nonsingular M-matrix if $\alpha > \rho(N)$ and a singular M-matrix if $\alpha = \rho(N)$.

2 A New Iterative Scheme

2.1 Preliminary

The equation (1) is a special case of the general nonsymmetric matrix Riccati equation of the form

$$-DX - XA + XSX + Q = 0, \tag{3}$$

where D, Q, S and A are real matrices of dimensions $m \times m$, $m \times n$, $n \times m$ and $n \times n$, respectively. The block matrix $K = \begin{pmatrix} A & -S \\ -Q & D \end{pmatrix}$ is an M-matrix. The general nonsymmetric matrix Riccati equation associated with M-matrices has many applications - in the Markov chains [7], in the transport theory [8] and many others. Nonsymmetric Riccati equation (1) arises from the game theory and more specially from the investigation of the open-loop Nash linear quadratic differential game [2, 1, 12, 13]. Research on the theories and the efficient numerical methods of (3) and it special case (1) has become a hot topic in recent years.

A more general problem on connected to the properties of the stabilising solution of the game theoretic algebraic Riccati equation is investigated in [4, 5, 11]. The solution of practical interest is the stabilizing nonnegative solution of (1).

There are many numerical methods up to now proposed for the minimal nonnegative solution of (3) with a nonsingular M-matrix. An effective method called alternately linearized implicit iteration method (ALI) was proposed and investigated in [3, 14, 6]. A new alternately linearized implicit iteration method (NALI) for computing the minimal nonnegative solution of (3) is introduced in [6]. The approach used in [14] is extended in the current paper and is applied to compute the stabilizing solution of (1).

2.2 Defining the NLIDI

In [10], we have proposed the following recursive equations (ALIDI) for computing the stabilizing solution of (1):

$$Y_i^{(k)}(\mu I_n + A - S_1 X_1^{(k)} - S_2 X_2^{(k)}) = (\mu I - A^T)X_i^{(k)} - Q_i, \quad i = 1, 2, \tag{4}$$

$$(\mu I_n + A^T - Y_i^{(k)} S_i)X_i^{(k+1)} = Y_i^{(k)}(\mu I - A + S_j X_j^{(k)}) - Q_i, \quad i, j = 1, 2, \ j \neq i\,, \tag{5}$$

$X_1^{(0)} = X_2^{(0)} = 0,\ k = 0, 1, 2, \ldots\,,\ \gamma < 0\,.$

Here, we propose another iteration to solve the same problem:

$$Y_i^{(k)}(\mu I_n + A - S_1 X_1^{(k)} - S_2 X_2^{(k)}) = (\mu I - A^T)X_i^{(k)} - Q_i\,, \tag{6}$$

$$(\mu I_n + A^T)X_i^{(k+1)} = Y_i^{(k)}(\mu I - A + S_1 Y_1^{(k)} + S_2 Y_2^{(k)}) - Q_i\,, \tag{7}$$

$i = 1, 2,\ X_1^{(0)} = X_2^{(0)} = 0,\ k = 0, 1, 2, \ldots,\ \mu < 0\,.$ We call it the NLIDI. Note that the first recursive equation (6) is the same as (4).

We formulate two statements for nonnegative matrices.

Lemma 1 *The following statements are equivalent for a Z-matrix $(-W)$:*

(a) $-W$ *is a nonsingular M-matrix.*

(b) $(\theta I_n - W)$ *is a nonsingular M-matrix, where* $\theta < 0$ *and* I_n *is the* $n \times n$ *unit matrix.*

(c) $W^{-1} \leq 0$ *(in elementwise order).*

(d) All eigenvalues of W *have negative real parts, i.e.* W *is asymptotically stable.*

Lemma 2 ([9]) *Let* $A = (a_{ij})$ *be an* $n \times n$ *M-matrix. If the elements of* $B = (b_{ij})$ *satisfy the relations:*

$$a_{ii} \geq b_{ii}, \quad (a_{ij}) \leq (b_{ij}) \leq 0,\ i \neq j,\ i, j = 1, \ldots, n\,,$$

then B *is also an M-matrix.*

We rewrite the matrix function $\mathfrak{R}(\mathcal{X})$ in the form $\mathfrak{R}(\mathcal{X}) = \begin{pmatrix} \mathcal{R}_1(X_1, X_2) \\ \mathcal{R}_2(X_1, X_2) \end{pmatrix}$, where

$$\mathcal{R}_1(X_1, X_2) = -A^T X_1 - X_1 A + X_1 S_1 X_1 + X_1 S_2 X_2 - Q_1,$$

$$\mathcal{R}_2(X_1, X_2) = -A^T X_2 - X_2 A + X_2 S_1 X_1 + X_2 S_2 X_2 - Q_2.$$

The equation $\mathfrak{R}(\mathcal{X}) = 0$ is equivalent to the set of Riccati equations $\mathcal{R}_1(X_1, X_2) = 0$, $\mathcal{R}_2(X_1, X_2) = 0$. We derive some properties and identities of the matrix functions $\mathcal{R}_1(.)$, $\mathcal{R}_2(.)$.

Lemma 3 *We construct the matrix sequences* $\{X_1{}^{(k)}, X_2{}^{(k)}, Y_1{}^{(k)}, Y_2{}^{(k)}\}_{k=0}^{\infty}$ *using (6)–(7) with initial values* $X_1^{(0)} = X_2^{(0)} = 0$*. The following properties hold:*

(i) $\mathcal{R}_i({X_1}^{(k)}, {X_2}^{(k)}) = (Y_i^{(k)} - X_i^{(k)})(\mu I_n + A - S_1 X_1^{(k)} - S_2 X_2^{(k)}), \quad i = 1, 2.$

(ii) $\mathcal{R}_i({Y_1}^{(k)}, {Y_2}^{(k)}) = (\mu I_n - A^T + Y_i^{(k)} S_i)(Y_i^{(k)} - X_i^{(k)}) + Y_i^{(k)} S_j (Y_j^{(k)} - X_j^{(k)}), \; i, j = 1, 2\,, \; j \neq i.$

(iii) $\mathcal{R}_i({Y_1}^{(k)}, {Y_2}^{(k)}) = (\mu I_n + A^T)(X_i^{(k+1)} - Y_i^{(k)}), \quad i = 1, 2.$

(iv) $\mathcal{R}_i({X_1}^{(k+1)}, {X_2}^{(k+1)}) = (X_i^{(k+1)} - Y_i^{(k)})(\mu I_n - A + S_i X_i^{(k+1)} + S_2 X_2^{(k+1)}) + Y_i^{(k)} S_1 (X_i^{(k+1)} - Y_i^{(k)}) + Y_i^{(k)} S_j (X_j^{(k+1)} - Y_j^{(k)}), \; i = 1, 2\,.$

In addition, the following equalities are true for any symmetric nonnegative matrices $\hat{X}_1, \hat{X}_2$*:*

(v) $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) = (Y_i^{(k)} - \hat{X}_i)(\mu I_n + A - S_1 X_1^{(k)} - S_2 X_2^{(k)}) - (\mu I_n - A + \hat{X}_i S_i)(X_i^{(k)} - \hat{X}_i) - \hat{X}_i S_j (X_j^{(k)} - \hat{X}_j),$ $i, j = 1, 2\,, \; j \neq i\,.$

(vi) $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) = (\mu I_n + A^T)(X_i^{(k+1)} - \hat{X}_i) - (Y_i^{(k)} - \hat{X}_i)(\mu I - A + S_1 Y_1^{(k)} + S_2 Y_2^{(k)}) - \hat{X}_i S_i (Y_i^{(k)} - \hat{X}_i) - \hat{X}_i S_j (Y_j^{(k)} - \hat{X}_j), \quad i, j = 1, 2\,, \; j \neq i\,.$

Proof. The proof is completed by a direct calculation. ■

2.3 Convergence Properties of the Matrix Sequences $\{X_i^{(k)}, Y_i^{(k)}\}_{k=0}^{\infty}$

Lemma 4 *Assume the matrix* $(-A)$ *is an M-matrix and* $\mathcal{Q} \geq 0$ *and* $\mathcal{S} < 0$, $\mu < 0$ *, such that* $(-\mu I - A)$ *is an M-matrix and* $(\mu I - A)$ *is nonpositive. Assume there exist symmetric nonnegative matrices* $\hat{X}_1$, $\hat{X}_2$ *such that* $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \geq 0$, $i = 1, 2$ *and* $-A + S_1 \hat{X}_1 + S_2 \hat{X}_2$ *is an M-matrix. We construct the matrix sequences* $\{{X_1}^{(k)}, {X_2}^{(k)}, {Y_1}^{(k)}, {Y_2}^{(k)}\}_{k=0}^{\infty}$ *using (6)–(7) with initial values* $X_1^{(0)} = X_2^{(0)} = 0$*. The following properties are satisfied:*

(a) $0 \leq Y_i^{(k)} \leq \hat{X}_i$ *for* $i = 1, 2, \; k = 0, 1, \ldots .$

(b) $0 \leq X_i^{(k)} \leq \hat{X}_i$ *for* $i = 1, 2, \; k = 0, 1, \ldots .$

The comparison is in the elementwise order.

Proof. Note that from lemma's assumptions we obtain the following inequalities $A^{-1} \leq 0, \; (\mu I + A)^{-1} \leq 0,$ $(\mu I + A^T)^{-1} \leq 0$ which we will use in the proof. For $k = 0$, we have $Y_i^{(0)} = -Q_i\, (\mu I_n + A)^{-1} \geq 0$, because $(\mu I_n + A)^{-1}$ is nonpositive. Using Lemma 3(v) we get

$$(Y_i^{(0)} - \hat{X}_i) = [\mathcal{R}_i(\hat{X}_1, \hat{X}_2) + (\mu I_n - A)(-\hat{X}_i) - \hat{X}_i S_i \hat{X}_i - \hat{X}_i S_j \hat{X}_j](\mu I_n + A)^{-1}\,,$$

with $i, j = 1, 2, \; j \neq i$. Note that $\mu I_n + A$, S_1, S_2 are nonpositive and thus $(Y_i^{(0)} - \hat{X}_i) \leq 0$ and $Y_i^{(0)} \leq \hat{X}_i$, $i = 1, 2$.

Further on, we apply equality (vi) from Lemma 3 for $k = 0$. The matrix $\mu I_n - A + S_1 Y_1^{(0)} + S_2 Y_2^{(0)}$ is nonpositive. We obtain

$$\begin{aligned} 0 &\leq \mathcal{R}_i(\hat{X}_1, \hat{X}_2) + \hat{X}_i S_i (Y_i^{(0)} - \hat{X}_i) + \hat{X}_i S_j (Y_j^{(0)} - \hat{X}_j) + (Y_i^{(0)} - \hat{X}_i)(\mu I - A + S_1 Y_1^{(0)} + S_2 Y_2^{(0)}) \\ &= (\mu I_n + A^T)(X_i^{(1)} - \hat{X}_i), \quad i, j = 1, 2, \; j \neq i. \end{aligned}$$

It follows that $X_i^{(1)} - \hat{X}_i \leq 0$, $i = 1, 2$. The statements (a) and (b) are proved for $k = 0$.

We assume that the inequalities are true: $0 \leq Y_i^{(r-1)} \leq \hat{X}_i$, $i = 1, 2$, and $0 \leq X_i^{(r)} \leq \hat{X}_i$, $i = 1, 2$, for $k = 0, 1, \ldots, r$.

We will prove the inequalities: $0 \le Y_i^{(r)} \le \hat{X}_i$, $0 \le X_i^{(r+1)} \le \hat{X}_i$, $i = 1,2$. We compute $Y_i^{(r)}$, $i = 1,2$ using (6) with $k = r$, $i = 1,2$. We have

$$Y_i^{(r)}(\mu I_n + A - S_1 X_1^{(r)} - S_2 X_2^{(r)}) = W_i^{(r)},$$

where

$$W_i^{(r)} := (\mu I - A^T) X_i^{(r)} - Q_i \le 0, \quad i = 1,2.$$

The inequality is true

$$S_1 \hat{X}_1 + S_2 \hat{X}_2 \le S_1 X_1^{(r)} + S_2 X_2^{(r)},$$

because $X_i^{(r)} \le \hat{X}_i$, $i = 1,2$. Thus, $-A + S_1\hat{X}_1 + S_2\hat{X}_2$ and $-\mu I - A + S_1\hat{X}_1 + S_2\hat{X}_2$ are M-matrices, and moreover $-\mu I - A + S_1 X_1^{(r)} + S_2 X_2^{(r)} = -Z^{(r)}$ is an M-matrix. Then $(Z^{(r)})^{-1} \le 0$. Therefore, $Y_i^{(r)} = W_i^{(r)}(Z^{(r)})^{-1} \ge 0$, $i = 1,2$.

According to Lemma 3(v) with $k = r$, we get

$$\begin{aligned} & \mathcal{R}_i(\hat{X}_1, \hat{X}_2) + (\mu I_n - A + \hat{X}_i S_i)(X_i^{(r)} - \hat{X}_i) + \hat{X}_i S_j (X_j^{(r)} - \hat{X}_j) \\ = \; & (Y_i^{(r)} - \hat{X}_i)(\gamma I_n + A - S_1 X_1^{(r)} - S_2 X_2^{(r)}), \quad i,j = 1,2, \; j \ne i. \end{aligned}$$

The left hand side of the above equality is nonnegative. We conclude $Y_i^{(r)} - \hat{X}_i \le 0$, $i = 1,2$.

Further on, we compute $X_i^{(r+1)}$, $i = 1,2$ using (7) with $k = r$. We obtain:

$$(\mu I_n + A^T) X_i^{(r+1)} = Y_i^{(r)}(\mu I - A + S_1 Y_1^{(r)} + S_2 Y_2^{(r)}) - Q_i\,,$$

$i = 1,2,$. Since the matrix $(\mu I - A + S_1 Y_1^{(r)} + S_2 Y_2^{(r)})$ is nonpositive the right hand side of the above equality is nonpositive and thus

$$X_i^{(r+1)} = (\mu I_n + A^T)^{-1}[Y_i^{(r)}(\mu I - A + S_1 Y_1^{(r)} + S_2 Y_2^{(r)}) - Q_i] \ge 0\,,$$

$i = 1,2$.

According to Lemma 3(vi) with $k = r$ we get

$$\begin{aligned} (\mu I_n + A^T)(X_i^{(r+1)} - \hat{X}_i) \;=\; & \mathcal{R}_i(\hat{X}_1, \hat{X}_2) + (Y_i^{(r)} - \hat{X}_i)(\mu I - A + S_1 Y_1^{(r)} + S_2 Y_2^{(r)}) \\ & + \hat{X}_i\, S_i (Y_i^{(r)} - \hat{X}_i) + \hat{X}_i S_j (Y_j^{(r)} - \hat{X}_j), \quad i,j = 1,2, \; j \ne i\,. \end{aligned}$$

The right hand side of the above equality is nonnegative and $(\mu I_n + A^T)^{-1} \le 0$. Thus $X_i^{(r+1)} - \hat{X}_i \le 0$, $i = 1,2$.

We compute $Y_i^{(r+1)}$, $i = 1,2$ using (6) with $k = r+1$, $i = 1,2$. We have

$$Y_i^{(r+1)}(\mu I_n + A - S_1 X_1^{(r+1)} - S_2 X_2^{(r+1)}) = W_i^{(r+1)},$$

where

$$W_i^{(r+1)} := (\mu I - A^T) X_i^{(r+1)} - Q_i \le 0, \quad i = 1,2.$$

Therefore the statements (a) and (b) are proved of $k = r+1$. This ends the proof. ■

In the next theorem we derive a sufficient condition for the convergence of the introduced NLIDI.

Theorem 1 *Assume the matrix $-A$ is an M-matrix and $\mathcal{Q} \ge 0$, and $\mathcal{S} \le 0$, $\mu < 0$ such that $(-\mu I - A)$ is an M-matrix and $\mu I - A$ is nonpositive. Assume there exist symmetric nonnegative matrices $\hat{X}_1$, $\hat{X}_2$, such that $\mathcal{R}_i(\hat{X}_1, \hat{X}_2) \ge 0$, $i = 1,2$. The matrix sequences $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ defined by (6)–(7) satisfy the following properties:*

(i) $\hat{X}_i \geq X_i^{(k+1)} \geq Y_i^{(k)} \geq X_i^{(k)}$ *for* $i = 1, 2,\ k = 0, 1, \ldots.$

(ii) $\mathcal{R}_i(X_1^{(k)}, X_2^{(k)}) \leq 0,\ \mathcal{R}_i(Y_1^{(k)}, Y_2^{(k)}) \leq 0,\ \mathcal{R}_i(X_1^{(k+1)}, X_2^{(k+1)}) \leq 0,\ i = 1, 2,\ k = 0, 1, \ldots.$

(iii) The matrix sequences $\{X_1^{(k)}, X_2^{(k)}\}_{k=0}^{\infty}$ *converge to the nonnegative minimal solution* $\tilde{X}_1$, $\tilde{X}_2$ *to the set of Riccati equations* $\mathcal{R}_1(X_1, X_2)) = 0$, $\mathcal{R}_2(X_1, X_2) = 0$ *with* $\tilde{X}_i \leq \hat{X}_i$.

(iv) Moreover, if $-A + \mathcal{S}\,\hat{\mathcal{X}}$ *and* $-D + \hat{\mathcal{X}}\,\mathcal{S}$ *are M-matrices, then the solution* $\tilde{\mathcal{X}} = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$ *is a left- right stabilizing solution of the nonsymmetric Nash Riccati equation* $\mathfrak{R}(\mathcal{X}) = 0$.

Proof. We construct the matrix sequences $\{{X_1}^{(k)}, {X_2}^{(k)}, {Y_1}^{(k)}, {Y_2}^{(k)}\}_{k=0}^{\infty}$ applying recursive equations (6)–(7) with $X_1^{(0)} = 0$, $X_2^{(0)} = 0$ and $\mu < 0$. The matrix $\mu I_n - A^T$ is nonpositive. According to Lemma 4 we know that the statements (a) and (b) are true. We have to prove the inequalities $X_i^{(k+1)} \geq Y_i^{(k)} \geq X_i^{(k)}$ for $i = 1, 2,\ k = 0, 1, \ldots.$ For $k = 0$ we have $Y_i^{(0)} \geq X_i^{(0)} = 0$, $i = 1, 2$. We have $\mathcal{R}_i(X_1^{(0)}, X_2^{(0)}) = -Q_i \leq 0$, $i = 1, 2$. Applying Lemma 3(ii), we get

$$\mathcal{R}_i({Y_1}^{(0)}, {Y_2}^{(0)}) = (\gamma I_n - A^T + Y_i^{(0)} S_i)(Y_i^{(0)}) + Y_i^{(0)} S_j (Y_j^{(0)}),\quad i, j = 1, 2,\ j \neq i\,.$$

We apply equalities (ii) and (iii) ($k = 0$) from Lemma 3

$$\mathcal{R}_i({Y_1}^{(0)}, {Y_2}^{(0)}) = (\mu I_n - A^T + Y_i^{(0)} S_i) Y_i^{(0)} + Y_i^{(0)} S_j Y_j^{(0)} \leq 0,$$

for $i, j = 1, 2,\ j \neq i$ and

$$\mathcal{R}_i({Y_1}^{(0)}, {Y_2}^{(0)}) = (\mu I_n + A^T)(X_i^{(1)} - Y_i^{(0)}),\quad i = 1, 2\,.$$

We obtain

$$(\mu I_n + A^T)(X_i^{(1)} - Y_i^{(0)}) = (\mu I_n - A^T + Y_i^{(0)} S_i)\, Y_i^{(0)} + Y_i^{(0)} S_j Y_j^{(0)} \leq 0\,,$$

$i, j = 1, 2,\ j \neq i$. Thus $X_i^{(1)} - Y_i^{(0)} \geq 0$, $i = 1, 2$.

Further on, we compute $X_1^{(1)}$, $X_2^{(1)}$ applying the recursive equation (7). According to Lemma 3(iv) we induce

$$\begin{aligned} \mathcal{R}_i({X_1}^{(1)}, {X_2}^{(1)}) &= (X_i^{(1)} - Y_i^{(0)})(\mu I_n - A + S_1 X_1^{(1)} + S_2 X_2^{(1)}) \\ &\quad + Y_i^{(0)} S_i (X_i^{(1)} - Y_i^{(0)}) + Y_i^{(0)} S_j (X_j^{(1)} - Y_j^{(0)}) \leq 0\,, \end{aligned}$$

$i, j = 1, 2,\ j \neq i$, because the matrices $\mu I_n - A$, $S_1 X_1^{(1)}$, $S_2 X_2^{(1)}$ are nonpositive.

Assume that the inequalities (i)–(ii) hold for $k = 0, 1, \ldots, r$. We know

$$X_i^{(r+1)} \geq Y_i^{(r)} \geq X_i^{(r)}\,,\quad i = 1, 2,$$

and

$$\mathcal{R}_i(X_1^{(r)}, X_2^{(r)}) \leq 0,\quad \mathcal{R}_i(Y_1^{(r)}, Y_2^{(r)}) \leq 0,\quad \mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+1)}) \leq 0,\quad i = 1, 2\,.$$

Applying Lemma 3(iv) with $k = r$, we get

$$\begin{aligned} \mathcal{R}_i({X_1}^{(r+1)}, {X_2}^{(r+1)}) &= (X_i^{(r+1)} - Y_i^{(r)})(\mu I_n - A + S_i X_i^{(r+1)} + S_2 X_2^{(r+1)}) + Y_i^{(r)} S_1 (X_i^{(r+1)} - Y_i^{(r)}) \\ &\quad + Y_i^{(r)} S_j (X_j^{(r+1)} - Y_j^{(r)}),\quad i = 1, 2\,. \end{aligned}$$

We know that $\mathcal{R}_i(X_1^{(r+1)}, X_2^{(r+1)}) \leq 0$, $i = 1, 2$ by the induction assumption and $(\mu I_n + A - S_1 X_1^{(r+1)} - S_2 X_2^{(r+1)})^{-1} \leq 0$. Thus, $Y_i^{(r+1)} \geq X_i^{(r+1)}$, $i = 1, 2$. Applying Lemma 3(ii) and the fact that $\mu I_n - A^T + Y_i^{(r+1)} S_i$, $i = 1, 2$ is a nonpositive matrix, we conclude

$$\mathcal{R}_i({Y_1}^{(r+1)}, {Y_2}^{(r+1)}) = (\mu I_n - A^T + Y_i^{(r+1)} S_i)(Y_i^{(r+1)} - X_i^{(r+1)}) + Y_i^{(r+1)} S_j (Y_j^{(r+1)} - X_j^{(r+1)}) \leq 0,$$

$i,j=1,2,\ j\neq i$.

According to Lemma 3(i) we extract $(i=1,2)$

$$\mathcal{R}_i(X_1^{(r+1)},X_2^{(r+1)}) = (Y_i^{(r+1)}-X_i^{(r+1)})(\mu I_n + A - S_1X_1^{(r+1)} - S_2X_2^{(r+1)}),$$

$$\mathcal{R}_i(X_1^{(r+1)},X_2^{(r+1)}) = (Y_i^{(r+1)}-X_i^{(r+1)})\,Z^{(r+1)},$$

$$\mathcal{R}_i(X_1^{(r+1)},X_2^{(r+1)})(Z^{(r+1)})^{-1} = (Y_i^{(r+1)}-X_i^{(r+1)}) \geq 0.$$

Since $(Z^{(r+1)})^{-1}\leq 0$, because $-Z^{(r+1)}$ is a nonsingular M-matrix, we infer

$$\mathcal{R}_i(X_1^{(r+1)},X_2^{(r+1)}) \leq 0, \quad i=1,2.$$

Further on, we compute $X_1^{(r+2)}$, $X_2^{(r+2)}$ applying the recursive equations (6)–(7). We know $\hat{X}_i \geq X_i^{(r+2)} \geq 0$, $i=1,2$.

We apply equalities (ii) and (iii) from Lemma 3 in order to obtain:

$$(\mu I_n + A^T)(X_i^{(r+2)} - Y_i^{(r+1)}) = \ (\mu I_n - A^T + Y_i^{(r+1)}S_i)(Y_i^{(r+1)} - X_i^{(r+1)}) + Y_i^{(r+1)}S_j(Y_j^{(r+1)} - X_j^{(r+1)}),$$

$i,j=1,2,\ j\neq i$. The right hand side of the above equality is nonposivive. The matrix $-\mu I_n - A^T + Y_i^{(r+1)}S_i$ is a nonsingular M-matrix, $i=1,2$. Thus $X_i^{(r+2)} - Y_i^{(r+1)} \geq 0$, $i=1,2$.

According to Lemma 3(iv) we write down

$$\begin{aligned}\mathcal{R}_i({X_1}^{(r+2)},{X_2}^{(r+2)}) \quad &= \quad (X_i^{(r+2)} - Y_i^{(r+1)})(\mu I_n - A + S_iX_i^{(r+2)} + S_2X_2^{(r+2)}) + Y_i^{(r+1)}S_1(X_i^{(r+2)} - Y_i^{(r+1)})\\ &\quad + Y_i^{(r+1)}S_j(X_j^{(r+2)} - Y_j^{(r+1)}), \quad i=1,2.\end{aligned}$$

and therefore $\mathcal{R}_i(X_1^{(r+2)},X_2^{(r+2)}) \leq 0$, $i=1,2$.

Hence, the induction process has been completed. Thus the matrix sequences $\{X_1^{(k)},\ X_2^{(k)}\}_{k=0}^{\infty}$ are nonnegative, monotonically increasing and bounded from above by $(\hat{X}_1,\hat{X}_2)$ (in the elementwise ordering). We denote $\lim_{k\to\infty}(\ X_1^{(k)},\ X_2^{(k)}) = (\tilde{X}_1,\tilde{X}_2)$. By taking the limits in (6)–(7) it follows that $(\tilde{X}_1,\tilde{X}_2)$ is a solution of $\mathcal{R}_i(X_1,X_2)=0$, $i=1,2$ with the property $\tilde{X}_i \leq \hat{X}_i$, $i=1,2$.

Further on, combining the facts that $-A+\mathcal{S}\,\tilde{\mathcal{X}}$ and $-D+\tilde{\mathcal{X}}\,\mathcal{S}$ are Z-matrices with Lemma 2 we consequently conclude $-A+\mathcal{S}\,\tilde{\mathcal{X}}$ and $-D+\tilde{\mathcal{X}}\,\mathcal{S}$ are also M-matrices, and $A-\mathcal{S}\,\tilde{\mathcal{X}}$ and $D-\tilde{\mathcal{X}}\,\mathcal{S}$ are asymptotically stable ones. Thus, the solution $\tilde{\mathcal{X}}$ is a left-right stabilizing solution of $\mathfrak{R}(\mathcal{X})=0$.

The proof is complete. ■

3 Numerical Examples

We consider a two-players game and we apply iterative methods ALIDI (4)–(5) and NLIDI (6)–(7) on two numerical examples. We have compared via numerical experiments the Newton method and the ALIDI method for computing the stabilizing solution of (1). The matrix coefficients A, B_i, Q_i and R_{ii} for $i=1,2$ are defined using the Matlab description. The numerical experiments are constructed following the approach applied in [11].

Example 1 *The matrix coefficients of (1) are*

$$A = \begin{pmatrix} -2.74 & 0.06 & 0.015 & 0.099\\ 0.2 & -2.5 & 0.064 & 0.08\\ 0.004 & 0.15 & -2.56 & 0.09\\ 0.14 & 0.12 & 0.21 & -2.57 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.5938\\ 0.2985\\ 0.49\\ 0.98 \end{pmatrix},$$

Table 1: Comparison between iterations with tol=1.0e-12 .

	ALIDI (4)–(5)		NLIDI (6)–(7)	
μ	avIt	CPU	avIt	CPU
-5	402	2.67s	431	2.7s
-3	256	1.76s	278	1.78s
-1	112	0.82s	112	0.79s
-0.5	40	0.37s	39	0.34s
-0.25	80	0.62s	77	0.57s

Table 2: Comparison between iterations with tol=1.0e-12, $\mu = -1$ and different values of n.

	ALIDI (4)–(5)			NLIDI (6)–(7)		
n	maxIt	avIt	CPU	maxIt	avIt	CPU
28	51	48.3	3.5s	50	47.5	2.2s
48	66	63.5	10.1s	66	62.9	8.5s
56	71	69.6	14.1s	72	68.7	11.9s
96	103	99.6	94.3	108	98.5	87.7s

$$B_2 = \begin{pmatrix} 2.8 & 0 & 0 & 0 \\ 0 & 2.9 & 0 & 0 \\ 0 & 0 & 2.84 & 1.5 \\ 0 & 0 & 1.5 & 1.3 \end{pmatrix},$$

$$Q_1 = eye(4,4)/2, \quad Q_1(1,1) = 2.0, \quad Q_1(4,4) = 1.5,$$

$$Q_2 = 0.5 * Q_1,$$

$$R_{11} = -1.909,$$

$$R_{22} = -eye(4,4), \quad R_{22}(1,1) = -50, \quad R_{22}(4,4) = -30.$$

We apply the above iterative methods for computing the stabilizing solution of (1) with the stop criteria $\|\mathcal{R}_i(X_1^{(k)}, X_2^{(k)})\| \le tol = 1.0e-12,\ i = 1,2$ *and different values of* μ*. It takes the following values:* $\mu = -5$, $\mu = -3$ *and* $\mu = -1$ *. Table 1 presents the computational results for different values of* μ*. The CPU time is computed for* 100 *runs for each value of* μ*.*

The iterations require the same number of iteration steps while finding the stabilising nonnegative solution of (1) for big values of $|\mu|$*. Yet, the conclusion is that the NLIDI method is more effective than ALIDI method for the lower values of* $|\mu|$ *under the conditions of Theorem 1.*

Example 2 *The matrix coefficients are*

$A = abs(randn(n))/9, \quad s = \max(abs(eig(A))) + 3.5,$ $\quad \mu$ *is a parameter with* $\mu < 0$,
for $i = 1:n,\ A(i,i) = -(A(i,i)) - s,$ *end*
$B_1 = abs(randn(n,4))/2,$
$B_2 = 0.7 * eye(n,n), \quad B_2(n,n) = 0.67,$
$Q_1 = zeros(n,n), \quad Q_1(1,1) = n/2, \quad Q_1(n,n) = 1.5,$
for $i = 1:n-1,\ Q_1(i,i+1) = 1/sqrt(n);\ Q_1(i+1,i) = 1/sqrt(n),$ *end*
$Q_2 = 2 * Q_1,$
$R_{11} = -10,$
$R_{22} = -eye(n,n), \quad R_{22}(1,1) = -50, \quad R_{22}(n,n) = -30.$

We are executing this example for different values of n, and 100 runs are completed for each value of n. We take $X_1^{(0)} = X_2^{(0)} = 0$ and thus $\mathcal{R}_i(X_1^{(0)}, X_2^{(0)}) = -Q_i \leq 0,\ i = 1, 2$, (i.e. the matrices are nonpositive).

Table 2 presents the computational results for different values of n.

Results from experiments, which are presented in Table 2, show that the numbers of iterations are slightly bigger than the ones in the ALIDI method. However the corresponding CPU times (for different values of n) for (6)–(7) are less than the corresponding CPU time for the ALIDI method.

4 Conclusion

The proposed NLIDI method (6)–(7) combine simplicity and efficient computer realization. It has the following advantages: (a) it is faster than the ALIDI method; (b) it uses only one matrix inverting for computing $X_1^{(k+1)}$, $X_2^{(k+1)}$, $k = 0, 1, \ldots$. The inverse matrix is computed only in the beginning of the iteration; (c) it is easy to extend the NLIDI iteration for a game with N players, $N > 2$. The advantages of the NLIDI are preserved in the case of more players; and (d) moreover, it is easy to reorganize the computations in the NLIDI in order to construct a parallel algorithm for computing the matrix sequences $X_1^{(k+1)}, \ldots, X_N^{(k+1)}$, $k = 0, 1, \ldots$.

The NLIDI iteration is more efficient on the basis of property (b). In fact the NLIDI iteration needs only one matrix inverting for computing all matrices of the matrix sequence $X_1^{(k+1)}, X_2^{(k+1)}$, $k = 0, 1, \ldots$.

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