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Relationships For Moments Of Progressively Type-II Right Censored Order Statistics From Bass Diffusion Model And Associated Inference*

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Abstract

In this paper we discuss Bass diffusion model, which is widely used in technology forecasting. Some recurrence relations between the single and product moments of progressively Type-II right censored order statistics from Bass diffusion model have been established. These relations would enable one to compute all the single and product moments of progressively Type-II right censored order statistics for all sample sizes n and all censoring schemes $(R_1, R_2, \dots, R_m)$, $m \le n$, in a simple recursive manner. For the estimation of the parameters and the reliability characteristic, maximum likelihood approach is used. Monte Carlo simulation study is conducted to compare the performance of the estimates for different censoring schemes.

1 Introduction

The Bass diffusion model was developed by Bass [6], and was described as the process of how new products get adapted as an interaction between users and potential users. He developed a growth model for the timing of first purchase (adoption) of a new product (innovation) by consumers in the marketplace, which has been widely used in marketing research both from a practical and theoretical point of view.

It has been described as one of the most famous empirical generalizations in marketing, along with the Dirichlet model of repeat buying and brand choice. The model is widely used in forecasting and technology forecasting. Mathematically, the Bass diffusion is a Riccati equation with constant coefficient.

The modelling and forecasting of the diffusion of innovations is a topic of increasing research interest in marketing science and other disciplines (cf. Meade and Islam [12]). In particular, the Bass diffusion model assumes that the time to first purchase of an innovation by a consumer can be modelled as a random variable, which henceforth we denote by X, with probability density function (p.d.f.) $f(x)$ satisfying the following Riccati differential equation with constant coefficients

$$\frac{f(x)}{1-F(x)} = p + qF(x), \quad x > 0,\ 0 < p \le 1,\ 0 \le q \le 1, \tag{1}$$

where $F(x) = P(X \le x)$, denotes the cumulative distribution function (c.d.f.) of X. The solution of the differential equation (1) gives the p.d.f. as

$$f(x) = \frac{((p+q)^2/p)e^{-(p+q)x}}{(1+(q/p)e^{-(p+q)x})^2}, \quad x > 0. \tag{2}$$

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The parameters p and q determine the shape of the diffusion process and are interpreted as the coefficients of innovation (external influence) and imitation (internal influence), respectively. If we denote sum of external influence and internal influence by α and the ratio of internal influence to external influence by β, then form of p.d.f. given by (2) becomes

$$f(x) = \frac{\alpha(1+\beta)e^{-\alpha x}}{(1+\beta e^{-\alpha x})^2}, \quad x > 0,\ \alpha > 0,\ \beta > -1, \tag{3}$$

where $\alpha = p + q$ and $\beta = q/p$. The corresponding c.d.f. is given by

$$F(x) = \frac{1 - e^{-\alpha x}}{1 + \beta e^{-\alpha x}}, \quad x > 0,\ \alpha > 0,\ \beta > -1. \tag{4}$$

From (4), the reliability function $R(t)$ is given as

$$R(t) = \frac{(1+\beta)e^{-\alpha t}}{1 + \beta e^{-\alpha t}}, \quad t > 0,\ \alpha > 0,\ \beta > -1. \tag{5}$$

The failure rate function of Bass diffusion model is given by

$$h(t) = \frac{f(t)}{R(t)} = \frac{\alpha}{1 + \beta e^{-\alpha t}}, \quad t > 0,\ \alpha > 0,\ \beta > -1. \tag{6}$$

It may be noted that the function $f(x)$ obtained in equation (3), is a well-defined probability density function when the parameters take the values $\alpha > 0$, $\beta > -1$, i.e.,

$$\int_0^\infty f(x)dx = 1.$$

From (3) and (4), we observe that the characterizing differential equation for the Bass diffusion model is given as

$$f(x) = \alpha\,[1 - F(x)] - \frac{\alpha\beta}{1+\beta}[1 - F(x)]^2. \tag{7}$$

Note. If imitation (internal influence) $q = 0$, then p.d.f. given in (2) or in (3) becomes the p.d.f. of exponential distribution.

2 Progressively Type-II Right Censored Order Statistics

Progressive censoring sampling scheme is very useful in reliability and life time studies. Its allowance for removal of live units from the test at various stages during the experiment will potentially save the experimenter cost while still allowing for the observation of some extreme data. Inferential issues based on this scheme have been extensively studied in the literature for a number of distributions by several authors including Aggarwala and Balakrishnan [1, 2], Balakrishnan and Aggarwala [4], Athar and Akhter [3], Cohen [7, 8, 9], Cohen and Whitten [10], Balakrishnan and Sandhu [5], Saran and Pushkarna [16, 17], Saran and Pande [15], Pushkarna et al. [13] , Saran et al. [14] and Singh and Khan [18].

Let the random variable X represent the waiting time of purchase of an innovation (new product). Suppose n independent individuals (namely $A_1, A_2, \ldots, A_n$) are observed for their respective purchases of a newly launched product, with continuous identically distributed purchase times $X_1, X_2, \ldots, X_n$. Suppose, further that a censoring scheme $R_1, R_2, \ldots, R_m$ is chosen before the experiment such that $n = m + \sum_{i=1}^{m} R_i$. Now immediately following the first purchase, suppose by A_k ($k \in [1, n]$), A_k and R_1 others (randomly chosen) i.e. $(R_1 + 1)$ individuals are removed from the experiment; immediately following the first purchase after that point. After second observed purchase say by A_p, A_p and R_2 others (randomly chosen) i.e. (R_2+1) individuals are removed from the experiment; this process continues until, at the time of the m^{th} observed purchase, R_m+1 individuals are removed from the test. Thus, in this type of sampling, we observe in all m observed purchases and

$$\sum_{i=1}^{m} R_i \text{ items are progressively censored so that } n = m + \sum_{i=1}^{m} R_i\,.$$

Let $X_{1:m:n}^{(R_1,R_2,\dots,R_m)} < X_{2:m:n}^{(R_1,R_2,\dots,R_m)} < \cdots < X_{m:m:n}^{(R_1,R_2,\dots,R_m)}$ be the m ordered observed first purchase times in a sample of size n from the Bass diffusion model as defined by (3), under the progressive Type-II right censoring scheme $(R_1,R_2,\dots,R_m)$, $m \le n$. Then the joint p.d.f. of

$$X_{1:m:n}^{(R_1,R_2,\dots,R_m)},\ X_{2:m:n}^{(R_1,R_2,\dots,R_m)},\dots,\ X_{m:m:n}^{(R_1,R_2,\dots,R_m)}$$

is given by (Balakrishnan and Sandhu [5])

$$f_{1,2,\dots,m:m:n}(x_1,x_2,\dots,x_m) = A(n,m-1)\prod_{i=1}^{m} f(x_i)[1-F(x_i)]^{R_i},$$
$$0 < x_1 < x_2 < \cdots < x_m < \infty, \tag{8}$$

where

$$A(n,m-1)=n(n-R_1-1)(n-R_1-R_2-2)\dots(n-R_1-R_2-\dots-R_{m-1}-m+1),$$

$f(x)$ and $F(x)$ are given by (3) and (4), respectively. Here, note that all the factors in $A(n,m-1)$ are positive integers. Also it may be observed that the different factors in $A(n,m-1)$ represent the number of units still on test immediately preceding the $1^{st}, 2^{nd}, \dots, m^{th}$ observed purchases, respectively.

Similarly, for convenience in notation, let us define for $q = 0,1,\dots,(p-1)$,

$$A(p,q) = p(p-R_1-1)(p-R_1-R_2-2)\dots(p-R_1-R_2-\cdots-R_q-q),$$

with all the factors being positive integers.

We shall denote the single moments of progressively Type-II right censored order statistics $X_{i:m:n}^{(R_1,R_2,\dots,R_m)}$, as follows:

For $1 \le i \le m \le n$, $k \ge 0$,

$$\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(k)}} = E\Big[X_{i:m:n}^{(R_1,R_2,\dots,R_m)}\Big]^k$$
$$= A(n,m-1)\iint\limits_{0<x_1<\cdots<x_m<\infty}\dots\int x_i^k\prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}\,dx_t, \tag{9}$$
$$\mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(1)}} \equiv \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)}\,.$$

We shall denote the product moments of progressively Type-II right censored order statistics as follows: For $1 \le i < j \le m \le n$, $r,s \ge 0$,

$$\mu_{i,j:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s)}} = E[\{X_{i:m:n}^{(R_1,R_2,\dots,R_m)}\}^r\{X_{j:m:n}^{(R_1,R_2,\dots,R_m)}\}^s]$$
$$= A(n,m-1)\iint\limits_{0<x_1<\cdots<x_m<\infty}\dots\int x_i^r x_j^s\prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t}\,dx_t, \tag{10}$$
$$\mu_{i,i:m:n}^{(R_1,R_2,\dots,R_m)^{(r,s)}} \equiv \mu_{i:m:n}^{(R_1,R_2,\dots,R_m)^{(r+s)}}, \quad 1 \le i \le m \le n\,. \tag{11}$$

In Sections 3 and 4, utilizing the characterizing differential equation (7), we have derived recurrence relations for the single and the product moments of progressively Type-II right censored order statistics from Bass diffusion model.

3 Recurrence Relations for Single Moments

Theorem 1 *For $2 \leq m \leq n$, $n \in N$ and for $k \geq 0$,*

$$\begin{aligned}\mu_{1:m:n+1}^{(R_1+1,R_2,\ldots,R_m)^{(k+1)}} &= \frac{(n+1)(1+\beta)}{n\alpha\beta(R_1+2)}\Bigg[\alpha(n-R_1-1)\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(k+1)}} \\ &\quad +\alpha(R_1+1)\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} \\ &\quad -\frac{\alpha\beta n(n-R_1-1)}{(1+\beta)(n+1)}\mu_{1:m-1:n+1}^{(R_1+R_2+2,R_3,\ldots,R_m)^{(k+1)}} \\ &\quad -(k+1)\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}\Bigg], \qquad (12)\end{aligned}$$

and for $m = 1$, $n \in N$ and $k \geq 0$,

$$\mu_{1:1:n+1}^{(n)^{(k+1)}} = \frac{(1+\beta)}{\alpha\beta}\left[\alpha\mu_{1:1:n}^{(n-1)^{(k+1)}} - \frac{(k+1)}{n}\mu_{1:1:n}^{(n-1)^{(k)}}\right]. \qquad (13)$$

Proof. Consider (9) for $i = 1$, i.e.,

$$\begin{aligned}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} &= A(n,m-1)\underset{0<x_2<x_3<\cdots<x_m<\infty}{\iint\cdots\int}\left\{\int_0^{x_2} x_1^k f(x_1)[1-F(x_1)]^{R_1}dx_1\right\} \\ &\quad \times f(x_2)[1-F(x_2)]^{R_2}\ldots f(x_m)[1-F(x_m)]^{R_m}dx_2dx_3\ldots dx_m \\ &= A(n,m-1)\underset{0<x_2<x_3<\cdots<x_m<\infty}{\iint\cdots\int} I(x_2)\prod_{t=2}^{m} f(x_t)[1-F(x_t)]^{R_t}dx_t, \qquad (14)\end{aligned}$$

where

$$I(x_2) = \int_0^{x_2} x_1^k f(x_1)[1-F(x_1)]^{R_1}dx_1.$$

Making use of the relation in (7) by replacing there in x by x_1, we have

$$I(x_2) = \alpha I_1(x_2) - \frac{\alpha\beta}{(1+\beta)}I_2(x_2), \qquad (15)$$

where

$$I_a(x_2) = \int_0^{x_2} x_1^k[1-F(x_1)]^{R_1+a}dx_1, \quad a = 1,2.$$

Integrating by parts yields,

$$I_a(x_2) = \frac{1}{(k+1)}\left[x_2^{k+1}[1-F(x_2)]^{R_1+a} + (R_1+a)\int_0^{x_2} x_1^{k+1}[1-F(x_1)]^{R_1+a-1}f(x_1)dx_1\right]. \qquad (16)$$

Substituting the values of $I_1(x_2)$ and $I_2(x_2)$ from (16) in (15) and then substituting the resultant expression for $I(x_2)$ in (14), we get

$$\begin{aligned}
&\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}\\
=\;& \frac{\alpha A(n,m-1)}{k+1}\underset{0<x_2<x_3<\cdots<x_m<\infty}{\iint\cdots\int} x_2^{k+1}f(x_2)[1-F(x_2)]^{R_1+R_2+1}dx_2\prod_{t=3}^{m}f(x_t)[1-F(x_t)]^{R_t}dx_t\\
&+\frac{\alpha A(n,m-1)(R_1+1)}{k+1}\underset{0<x_1<x_2<\cdots<x_m<\infty}{\iint\cdots\int} x_1^{k+1}\prod_{t=1}^{m}f(x_t)[1-F(x_t)]^{R_t}dx_t\\
&-\frac{\alpha\beta A(n,m-1)}{(1+\beta)(k+1)}\underset{0<x_2<x_3<\cdots<x_m<\infty}{\iint\cdots\int} x_2^{k+1}f(x_2)[1-F(x_2)]^{R_1+R_2+2}dx_2\prod_{t=3}^{m}f(x_t)[1-F(x_t)]^{R_t}dx_t\\
&-\frac{\alpha\beta A(n,m-1)(R_1+2)}{(1+\beta)(k+1)}\underset{0<x_1<x_2<\cdots<x_m<\infty}{\iint\cdots\int} x_1^{k+1}f(x_1)[1-F(x_1)]^{R_1+1}dx_1\\
&\times\prod_{t=2}^{m}f(x_t)[1-F(x_t)]^{R_t}dx_t\\
=\;&\frac{\alpha(R_1+1)}{k+1}\mu_{1:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}}+\frac{\alpha(n-R_1-1)}{k+1}\mu_{1:m-1:n}^{(R_1+R_2+1,R_3,\ldots,R_m)^{(k+1)}}\\
&-\frac{\alpha\beta n(n-R_1-1)}{(1+\beta)(n+1)(k+1)}\mu_{1:m-1:n+1}^{(R_1+R_2+2,R_3,\ldots,R_m)^{(k+1)}}-\frac{\alpha\beta n(R_1+2)}{(1+\beta)(n+1)(k+1)}\mu_{1:m:n+1}^{(R_1+1,R_2,\ldots,R_m)^{(k+1)}},
\end{aligned}$$

which on rearranging the terms leads to (12).

To prove the relation in (13), we take $i=1$, $m=1$ in (9) and then using (7) with x replaced by x_1, we get

$$\mu_{1:1:n}^{(R_1)^{(k)}}=A(n,0)\left[\alpha\int_0^\infty x_1^k[1-F(x_1)]^{R_1+1}dx_1-\frac{\alpha\beta}{(1+\beta)}\int_0^\infty x_1^k[1-F(x_1)]^{R_1+2}dx_1\right].$$

Integrating the right hand side integrals by parts and noting that $R_1=n-1$, since the equation $n=m+R_1+R_2+\cdots+R_m$ must be satisfied, we have

$$\begin{aligned}
\mu_{1:1:n}^{(n-1)^{(k)}} &= \frac{n}{k+1}\left[\alpha n\int_0^\infty x_1^{k+1}[1-F(x_1)]^{n-1}f(x_1)dx_1-\frac{\alpha\beta(n+1)}{(1+\beta)}\int_0^\infty x_1^{k+1}[1-F(x_1)]^{n}f(x_1)dx_1\right]\\
&= \frac{n}{k+1}\left[\alpha\mu_{1:1:n}^{(n-1)^{(k+1)}}-\frac{\alpha\beta}{(1+\beta)}\mu_{1:1:n+1}^{(n)^{(k+1)}}\right],
\end{aligned}$$

which on rearranging the terms leads to (13). ■

Remark 1 *It may be noted that the first progressively Type-II right censored order statistic* $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}$ *is the same as the first usual order statistic from a sample of size* n*, regardless of the censoring scheme employed. This is because no censoring has taken place before this time.*

Theorem 2 *For* $2\le i\le m-1$, $m\le n$ *and* $k\ge 0$,

$$\begin{aligned}
&\mu_{i:m:n+1}^{(R_1,\ldots,R_{i-1},R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}}\\
=\;&\frac{(1+\beta)A(n+1,i-1)}{\alpha\beta(R_i+2)A(n,i-1)}\Bigg[\alpha(n-R_1-R_2-\cdots-R_i-i)\mu_{i:m-1:n}^{(R_1,R_2,\ldots,R_{i-1},R_i+R_{i+1}+1,R_{i+2},\ldots,R_m)^{(k+1)}}\\
&-\alpha(n-R_1-R_2-\cdots-R_{i-1}-i+1)\mu_{i-1:m-1:n}^{(R_1,R_2,\ldots,R_{i-2},R_{i-1}+R_i+1,R_{i+1},\ldots,R_m)^{(k+1)}}\\
&+\alpha(R_i+1)\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}}-\frac{\alpha\beta A(n,i)}{(1+\beta)A(n+1,i-1)}\mu_{i:m-1:n+1}^{(R_1,R_2,\ldots,R_{i-1},R_i+R_{i+1}+2,R_{i+2},\ldots,R_m)^{(k+1)}}\\
&+\frac{\alpha\beta A(n,i-1)}{(1+\beta)A(n+1,i-2)}\mu_{i-1:m-1:n+1}^{(R_1,R_2,\ldots,R_{i-2},R_{i-1}+R_i+2,R_{i+1},\ldots,R_m)^{(k+1)}}-(k+1)\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}\Bigg]. \quad (17)
\end{aligned}$$

Proof. Using (9), we have for $2 \le i \le m-1$,

$$\mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}} = A(n,m-1) \underset{0<x_1<\cdots<x_{i-1}<x_{i+1}<\cdots<x_m<\infty}{\int \cdots \int \cdots \int} J(x_{i-1},x_{i+1}) \prod_{\substack{t=1 \\ t\neq i}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t, \quad (18)$$

where

$$J(x_{i-1},x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^k f(x_i)[1-F(x_i)]^{R_i} dx_i.$$

Making use of the relation in (7) and splitting the integral accordingly into two, we have

$$J(x_{i-1},x_{i+1}) = \alpha J_1(x_{i-1},x_{i+1}) - \frac{\alpha\beta}{(1+\beta)} J_2(x_{i-1},x_{i+1}), \quad (19)$$

where

$$J_a(x_{i-1},x_{i+1}) = \int_{x_{i-1}}^{x_{i+1}} x_i^k [1-F(x_i)]^{R_i+a} dx_i, \quad a = 1, 2.$$

Integrating by parts yields

$$\begin{aligned} J_a(x_{i-1},x_{i+1}) &= \frac{1}{(k+1)}\Bigg[x_{i+1}^{k+1}[1-F(x_{i+1})]^{R_i+a} - x_{i-1}^{k+1}[1-F(x_{i-1})]^{R_i+a} \\ &\quad +(R_i+a)\int_{x_{i-1}}^{x_{i+1}} x_i^{k+1}[1-F(x_i)]^{R_i+a-1} f(x_i) dx_i\Bigg]. \end{aligned} \quad (20)$$

Upon substituting for $J_1(x_{i-1},x_{i+1})$ and $J_2(x_{i-1},x_{i+1})$ from (20) in (19) and then substituting the resultant expression for $J(x_{i-1},x_{i+1})$ in (18) and simplifying, on using (9), it leads to (17). ∎

Corollary 3 *For $2 \le m \le n$, $n \in N$ and $k \ge 0$,*

$$\begin{aligned} &\mu_{m:m:n+1}^{(R_1,\ldots,R_{m-1},R_m+1)^{(k+1)}} \\ &= \frac{(1+\beta)A(n+1,m-1)}{\alpha\beta(R_m+2)A(n,m-1)}\Bigg[\frac{\alpha\beta A(n,m-1)}{(1+\beta)A(n+1,m-2)}\mu_{m-1:m-1:n+1}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+2)^{(k+1)}} \\ &\quad -\alpha(n-R_1-R_2-\cdots-R_{m-1}-m+1)\mu_{m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(k+1)}} \\ &\quad +\alpha(R_m+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k+1)}} - (k+1)\mu_{m:m:n}^{(R_1,R_2,\ldots,R_m)^{(k)}}\Bigg]. \end{aligned} \quad (21)$$

Proof. The recurrence relation in (21) can be established by following exactly the same steps as those used in proving Theorem 2. ∎

4 Recurrence Relations for Product Moments

The recurrence relations for the product moments, defined in equation (10), of progressively Type-II right censored order statistics from Bass Diffusion model, are given in the following theorems.

Theorem 4 *For $1 \leq i < j < m$, $m \leq n$ and $r, s \geq 0$,*

$$\begin{aligned}
&\mu_{i,j:m:n+1}^{(R_1,R_2,\ldots,R_{j-1},R_j+1,R_{j+1},\ldots,R_m)^{(r,s+1)}} \\
= \quad & \frac{(1+\beta)A(n+1,j-1)}{(R_j+2)\alpha\beta A(n,j-1)}\Big[\alpha(n-R_1-R_2-\cdots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)^{(r,s+1)}} \\
& -\alpha(n-R_1-R_2-\cdots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,R_2,\ldots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)^{(r,s+1)}} \\
& +\alpha(R_j+1)\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s+1)}} - \frac{\alpha\beta A(n,j)}{(1+\beta)A(n+1,j-1)}\mu_{i,j:m-1:n+1}^{(R_1,R_2,\ldots,R_{j-1},R_j+R_{j+1}+2,R_{j+2},\ldots,R_m)^{(r,s+1)}} \\
& +\frac{\alpha\beta A(n,j-1)}{(1+\beta)A(n+1,j-2)}\mu_{i,j-1:m-1:n+1}^{(R_1,R_2,\ldots,R_{j-2},R_{j-1}+R_j+2,R_{j+1},\ldots,R_m)^{(r,s+1)}} \\
& -(s+1)\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s)}}\Big]. \qquad (22)
\end{aligned}$$

Proof. From equation (10), we have

$$\begin{aligned}
&\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s)}} \\
= \quad & A(n,m-1) \int\cdots\int\cdots\int_{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty} x_i^r \left\{\int_{x_{j-1}}^{x_{j+1}} x_j^s f(x_j)\left[1-F(x_j)\right]^{R_j} dx_j\right\} \\
& \times \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)\left[1-F(x_t)\right]^{R_t} dx_t \\
= \quad & A(n,m-1) \int\cdots\int\cdots\int_{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty} x_i^r T(x_{j-1},x_{j+1}) \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t, \qquad (23)
\end{aligned}$$

where

$$T(x_{j-1},x_{j+1}) = \int_{x_{j-1}}^{x_{j+1}} x_j^s f(x_j)[1-F(x_j)]^{R_j} dx_j.$$

Making use of the relation in (7) and splitting the integral accordingly into two, we have

$$T(x_{j-1},x_{j+1}) = \alpha T_1(x_{j-1},x_{j+1}) - \frac{\alpha\beta}{(1+\beta)} T_2(x_{j-1},x_{j+1}), \qquad (24)$$

where

$$T_a(x_{j-1},x_{j+1}) = \int_{x_{j-1}}^{x_{j+1}} x_j^s[1-F(x_j)]^{R_j+a} dx_j, \quad a=1,2.$$

Integrating by parts yields

$$\begin{aligned}
T_a(x_{j-1},x_{j+1}) = \frac{1}{(s+1)}\Big[& x_{j+1}^{s+1}[1-F(x_{j+1})]^{R_j+a} - x_{j-1}^{s+1}[1-F(x_{j-1})]^{R_j+a} \\
& + (R_j+a)\int_{x_{j-1}}^{x_{j+1}} x_j^{s+1}[1-F(x_j)]^{R_j+a-1} f(x_j) dx_j\Big]. \qquad (25)
\end{aligned}$$

Upon substituting for $T_1(x_{j-1}, x_{j+1})$ and $T_2(x_{j-1}, x_{j+1})$ from (25) in (24) and then substituting the resultant expression for $T(x_{j-1}, T_{j+1})$ in (23) and simplifying, on using (10), we get

$$
\begin{aligned}
&\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s)}}\\
=\;& \frac{\alpha A(n,m-1)}{s+1} \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int\cdots\int\cdots\int} x_i^r x_{j+1}^{s+1}[1-F(x_{j+1})]^{R_j+1} \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t\\
&-\frac{\alpha A(n,m-1)}{s+1} \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int\cdots\int\cdots\int} x_i^r x_{j-1}^{s+1}[1-F(x_{j-1})]^{R_j+1} \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t\\
&+\frac{\alpha A(n,m-1)(R_j+1)}{s+1} \underset{0<x_1<\cdots<x_m<\infty}{\int\cdots\int\cdots\int} x_i^r x_j^{s+1} \prod_{t=1}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t\\
&-\frac{\alpha\beta A(n,m-1)}{(1+\beta)(s+1)} \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int\cdots\int\cdots\int} x_i^r x_{j+1}^{s+1}[1-F(x_{j+1})]^{R_j+2} \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t\\
&+\frac{\alpha\beta A(n,m-1)}{(1+\beta)(s+1)} \underset{0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_m<\infty}{\int\cdots\int\cdots\int} x_i^r x_{j-1}^{s+1}[1-F(x_{j-1})]^{R_j+2} \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t\\
&-\frac{\alpha\beta A(n,m-1)(R_j+2)}{(1+\beta)(s+1)} \underset{0<x_1<x_2<\cdots<x_m<\infty}{\int\cdots\int\cdots\int} x_i^r x_j^{s+1} f(x_j)[1-F(x_j)]^{R_j+1} dx_j \prod_{\substack{t=1\\t\neq j}}^{m} f(x_t)[1-F(x_t)]^{R_t} dx_t.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&(s+1)\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s)}}\\
&= \alpha(n-R_1-R_2-\cdots-R_j-j)\mu_{i,j:m-1:n}^{(R_1,R_2,\ldots,R_{j-1},R_j+R_{j+1}+1,R_{j+2},\ldots,R_m)^{(r,s+1)}}\\
&\quad-\alpha(n-R_1-R_2-\cdots-R_{j-1}-j+1)\mu_{i,j-1:m-1:n}^{(R_1,R_2,\ldots,R_{j-2},R_{j-1}+R_j+1,R_{j+1},\ldots,R_m)^{(r,s+1)}}\\
&\quad+\alpha(R_j+1)\mu_{i,j:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s+1)}} - \frac{\alpha\beta A(n,j)}{(1+\beta)A(n+1,j-1)}\\
&\quad\times \mu_{i,j:m-1:n+1}^{(R_1,R_2,\ldots,R_{j-1},R_j+R_{j+1}+2,R_{j+2},\ldots,R_m)^{(r,s+1)}} + \frac{\alpha\beta A(n,j-1)}{(1+\beta)A(n+1,j-2)}\\
&\quad\times \mu_{i,j-1:m-1:n+1}^{(R_1,R_2,\ldots,R_{j-2},R_{j-1}+R_j+2,R_{j+1},\ldots,R_m)^{(r,s+1)}} - \frac{(R_j+2)\alpha\beta A(n,j-1)}{(1+\beta)A(n+1,j-1)}\\
&\quad\times \mu_{i,j:m:n+1}^{(R_1,R_2,\ldots,R_{j-1},R_j+1,R_{j+1},\ldots,R_m)^{(r,s+1)}},
\end{aligned}
$$

which on rearranging the terms leads to (22). ■

Remark 2 *It may be noted that Theorem 4 holds even for $j = i+1$, without altering the proof, provided we realize that $\mu_{i,i:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s)}} = \mu_{i:m:n}^{(R_1,R_2,\ldots,R_m)^{(r+s)}}$, as mentioned in equation (11).*

Theorem 5 *For $1 \le i \le m-1$, $m \le n$ and $r, s \ge 0$,*

$$
\begin{aligned}
&\mu_{i,m:m:n+1}^{(R_1,R_2,\ldots,R_{m-1},R_m+1)^{(r,s+1)}}\\
=\;& \frac{(1+\beta)A(n+1,m-1)}{(R_m+2)\alpha\beta A(n,m-1)}\Bigg[\frac{\alpha\beta A(n,m-1)}{(1+\beta)A(n+1,m-2)}\mu_{i,m-1:m-1:n+1}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+2)^{(r,s+1)}}\\
&-\alpha(n-R_1-R_2-\cdots-R_{m-1}-m+1)\mu_{i,m-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},R_{m-1}+R_m+1)^{(r,s+1)}}\\
&+\alpha(R_m+1)\mu_{i,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s+1)}} - (s+1)\mu_{i,m:m:n}^{(R_1,R_2,\ldots,R_m)^{(r,s)}}\Bigg].
\end{aligned}
\tag{26}
$$

Proof. The relation in (26) may be proved by following exactly the same steps as those used in proving Theorem 4. ■

Remark 3 *For the special case $R_1 = R_2 = \cdots = R_m = 0$ so that $m = n$, in which case the progressively censored order statistics become the usual order statistics $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, whose single moments are denoted by $\mu_{i:n}^{(k)}$ for $1 \le i \le n$ and product moments are denoted by $\mu_{i,j:n}^{(r,s)}$ for $1 \le i < j \le n$, the recurrence relations established in Sections 3 and 4 reduce to that of usual order statistics from Bass diffusion model.*

5 Recursive Computational Algorithm

Thomas and Wilson [19] gave a computational method for obtaining single and product moments of progressively Type-II right censored order statistics from an arbitrary continuous distribution through a mixture form that expresses them in terms of those of the usual order statistics from a sample of size n. Utilizing the knowledge of recurrence relations obtained in Sections 3 and 4 in a systematic manner, along with the mixture formula for missing moments, one can evaluate the moments of progressively Type-II right censored order statistics from Bass diffusion model for all sample sizes and all censoring schemes $(R_1, R_2, \ldots, R_m)$ in a simple recursive way. The same has been demonstrated in the next section.

5.1 Single moments

Case I: When $n = 1$, then $m = 1$.

In this case, we have only one progressive censoring scheme $R_1 = 0$. Thus, from equation (9), we have

$$E(X_{1:1:1}^{(0)})^k = \mu_{1:1:1}^{(0)^{(k)}} = \mu_{1:1}^{(k)} = E(X^k) = -\frac{(1+\beta)\Gamma(k+1)}{\alpha^k \beta} L_{i_k}(-\beta), \tag{27}$$

where $L_{i_k}(z)$ is called the polylogarithm of order k, $k = 1, 2, \ldots$, defined as

$$L_{i_k}(z) = \int_0^z \frac{L_{i_{k-1}}(u)}{u} du, \text{ with } L_{i_0}(z) = \frac{z}{1-z}, \quad \text{where } k = 1, 2, \ldots .$$

The case $k = 1$ is the natural logarithm $L_{i_1}(z) = -\log(1-z)$ (cf. Jodŕa [11]).

Putting $k = 1$ and 2, in equation (27), we get

$$\mu_{1:1:1}^{(0)} = \mu_{1:1}^{(1)} = E(X) = \frac{(1+\beta)}{\alpha\beta} \log(1+\beta),$$

and

$$\mu_{1:1:1}^{(0)^{(2)}} = \mu_{1:1}^{(2)} = E(X^2) = -\frac{2(1+\beta)}{\alpha^2\beta} L_{i_2}(-\beta).$$

Proceeding in a similar manner $\mu_{1:1:1}^{(0)^{(k)}}$ for all $k = 1, 2, \ldots$, can be calculated.

Case II: When $n = 2$, then $m = 1$ or 2.

Subcase (i): $m = 1$

We have only one progressive censoring scheme $R_1 = 1$, and in this case we have from equation (13), on putting $n = 1$,

$$E\left(X_{1:1:2}^{(1)}\right)^{k+1} = \mu_{1:1:2}^{(1)^{(k+1)}} = \frac{(1+\beta)}{\alpha\beta}\left[\alpha\mu_{1:1:1}^{(0)^{(k+1)}} - (k+1)\mu_{1:1:1}^{(0)^{(k)}}\right]. \tag{28}$$

Putting $k = 0$ in equation (28), we get

$$\mu_{1:1:2}^{(1)^{(1)}} = \mu_{1:1:2}^{(1)} = \mu_{1:2}^{(1)} = \frac{(1+\beta)}{\alpha\beta}[\alpha\mu_{1:1:1}^{(0)} - 1]$$
$$= \frac{(1+\beta)}{\alpha\beta^2}[(1+\beta)\log(1+\beta) - \beta]. \quad (29)$$

Also, for $k = 1$ in equation (28), we have

$$\mu_{1:1:2}^{(1)^{(2)}} = \frac{(1+\beta)}{\alpha\beta}[\alpha\mu_{1:1:1}^{(0)^{(2)}} - 2\mu_{1:1:1}^{(0)}], \quad (30)$$

where

$$\mu_{1:1:1}^{(0)^{(2)}} \text{ and } \mu_{1:1:1}^{(0)} \text{ can be calculated using (27).}$$

Proceeding in a similar manner $\mu_{1:1:2}^{(1)^{(k)}}$ for all $k = 1, 2, \ldots,$ can be calculated.

Subcase (ii): $m = 2$

We have only one progressive censoring scheme $R_1 = R_2 = 0$, in this case we have

$$E(X_{1:2:2}^{(0,0)}) = \mu_{1:2:2}^{(0,0)} = \mu_{1:2} \text{ and } E(X_{2:2:2}^{(0,0)}) = \mu_{2:2:2}^{(0,0)} = \mu_{2:2}.$$

Also,

$$E(X_{1:2:2}^{(0,0)})^2 = \mu_{1:2:2}^{(0,0)^{(2)}} = \mu_{1:2}^{(2)} \text{ and } E(X_{2:2:2}^{(0,0)})^2 = \mu_{2:2:2}^{(0,0)^{(2)}} = \mu_{2:2}^{(2)}$$

and these values concerning ordinary order statistics can be evaluated.

Case III: When $n = 3$, then $m = 1$ or 2 or 3.

Subcase (i): $m = 1$

We have only one progressive censoring scheme $R_1 = 2$, and in this case we have from (13), on putting $k = 0$ and $n = 2$,

$$E(X_{1:1:3}^{(2)}) = \mu_{1:1:3}^{(2)} = \frac{(1+\beta)}{\alpha\beta}\left[\alpha\mu_{1:1:2}^{(1)} - \frac{1}{2}\right],$$

and upon taking $k = 1$ and $n = 2$, we get

$$E(X_{1:1:3}^{(2)})^2 = \mu_{1:1:3}^{(2)^{(2)}} = \frac{(1+\beta)}{\alpha\beta}[\alpha\mu_{1:1:2}^{(1)^{(2)}} - \mu_{1:1:2}^{(1)}],$$

where, $\mu_{1:1:2}^{(1)}$ and $\mu_{1:1:2}^{(1)^{(2)}}$ can be calculated using (29) and (30), respectively.

Proceeding in a similar manner $\mu_{1:1:3}^{(2)^{(k)}}$ for all $k = 1, 2, \ldots$ can be calculated.

Subcase (ii): $m = 2$

We have only two progressive censoring schemes. One is $R_1 = 1$ and $R_2 = 0$ and the other is $R_1 = 0$ and $R_2 = 1$.

When $R_1 = 1$ and $R_2 = 0$

On putting $k = 0$, $m = n = 2$, $R_1 = 0$ and $R_2 = 0$ in (12), we get

$$E(X_{1:2:3}^{(1,0)}) = \mu_{1:2:3}^{(1,0)} = \frac{3(1+\beta)}{4\alpha\beta}\left[\alpha\mu_{1:1:2}^{(1)} + \alpha\mu_{1:2}^{(1)} - \frac{2\alpha\beta}{3(1+\beta)}\mu_{1:1:3}^{(2)} - 1\right]$$

and upon taking $k = 1$, $m = n = 2$, $R_1 = 0$ and $R_2 = 0$ in (12), we get

$$E(X_{1:2:3}^{(1,0)})^2 = \mu_{1:2:3}^{(1,0)^{(2)}} = \frac{3(1+\beta)}{4\alpha\beta}\left[\alpha\mu_{1:1:2}^{(1)^{(2)}} + \alpha\mu_{1:2}^{(2)} - \frac{2\alpha\beta}{3(1+\beta)}\mu_{1:1:3}^{(2)^{(2)}} - 2\mu_{1:2}^{(1)}\right].$$

Further, on using mixture formula, we have

$$\mu_{2:2:3}^{(1,0)} = \frac{1}{2}\left[\mu_{2:3} + \mu_{3:3}\right] \quad \text{and} \quad \mu_{2:2:3}^{(1,0)^{(2)}} = \frac{1}{2}\left[\mu_{2:3}^{(2)} + \mu_{3:3}^{(2)}\right].$$

Proceeding in a similar manner $\mu_{1:2:3}^{(1,0)^{(k)}}$ and $\mu_{2:2:3}^{(1,0)^{(k)}}$ for all $k = 1, 2, \ldots$ can be calculated.

When $R_1 = 0$ and $R_2 = 1$

In this case, we find that

$$E(X_{1:2:3}^{(0,1)}) = \mu_{1:2:3}^{(0,1)} = \mu_{1:3}, \quad E(X_{2:2:3}^{(0,1)}) = \mu_{2:2:3}^{(0,1)} = \mu_{2:3},$$

$$E(X_{1:2:3}^{(0,1)})^2 = \mu_{1:2:3}^{(0,1)^{(2)}} = \mu_{1:3}^{(2)} \quad \text{and} \quad E(X_{2:2:3}^{(0,1)})^2 = \mu_{2:2:3}^{(0,1)^{(2)}} = \mu_{2:3}^{(2)}.$$

Other moments can similarly be obtained.

Subcase (iii): $m = 3$

We have only one progressive censoring scheme $R_1 = 0$, $R_2 = 0$ and $R_3 = 0$. In this case

$$E(X_{1:3:3}^{(0,0,0)}) = \mu_{1:3:3}^{(0,0,0)} = \mu_{1:3}, \quad E(X_{2:3:3}^{(0,0,0)}) = \mu_{2:3:3}^{(0,0,0)} = \mu_{2:3},$$

$$E(X_{1:3:3}^{(0,0,0)})^2 = \mu_{1:3:3}^{(0,0,0)^{(2)}} = \mu_{1:3}^{(2)}, \quad E(X_{2:3:3}^{(0,0,0)})^2 = \mu_{2:3:3}^{(0,0,0)^{(2)}} = \mu_{2:3}^{(2)},$$

and

$$E(X_{3:3:3}^{(0,0,0)})^2 = \mu_{3:3:3}^{(0,0,0)^{(2)}} = \mu_{3:3}^{(2)}.$$

All these values can be obtained by using the results of Jodŕa [11] for ordinary order statistics.

5.2 Product moments

Case I: When $n = 2$ and $m = 2$.

In this case we have only one progressive censoring scheme i.e. $R_1 = R_2 = 0$. Thus, from equation (10), we have

$$\begin{aligned} E(X_{1:2:2}^{(0,0)} X_{2:2:2}^{(0,0)}) &= \mu_{1,2:2:2}^{(0,0)} = \mu_{1:2:2} \\ &= (\mu_{1:1})^2 \\ &= \left(\frac{(1+\beta)}{\alpha\beta}\log(1+\beta)\right)^2. \end{aligned}$$

Case II: When $n = 3$ and $m = 2$.

We have only two progressive censoring schemes. One is $R_1 = 1$ and $R_2 = 0$ and the other is $R_1 = 0$ and $R_2 = 1$.

When $R_1 = 1$ and $R_2 = 0$

In this case we have $E(X_{1:2:3}^{(1,0)} X_{2:2:3}^{(1,0)}) = \mu_{1,2:2:3}^{(1,0)} = \frac{1}{2}(\mu_{1,2:3} + \mu_{1,3:3})$, from the mixture formula.

When $R_1 = 0$ and $R_2 = 1$

In this case we have $E(X_{1:2:3}^{(0,1)} X_{2:2:3}^{(0,1)}) = \mu_{1,2:2:3}^{(0,1)}$, which can be computed on putting $m = 2$, $n = 2$, $r = 1$, $s = 1$, $R_1 = 0$ and $R_2 = 0$ in (26).

Case III: $n = 3$ **and** $m = 3$.

In this case we have only one progressive censoring scheme $R_1 = R_2 = R_3 = 0$ and

$$E(X_{1:3:3}^{(0,0,0)} X_{2:3:3}^{(0,0,0)}) = \mu_{1,2:3:3}^{(0,0,0)} = \mu_{1,2:3}, \; E(X_{1:3:3}^{(0,0,0)} X_{3:3:3}^{(0,0,0)}) = \mu_{1,3:3:3}^{(0,0,0)} = \mu_{1,3:3}$$

and

$$E(X_{2:3:3}^{(0,0,0)} X_{3:3:3}^{(0,0,0)}) = \mu_{2,3:3:3}^{(0,0,0)} = \mu_{2,3:3}.$$

Likewise, one could proceed for higher values of n and all choices of m and $(R_1, R_2, \ldots, R_m)$.

6 Maximum Likelihood Estimators (MLEs)

Based on the observed sample $x_1 < x_2 < \cdots < x_m$ from a progressive Type-II censoring scheme, $(R_1, R_2, \ldots, R_m)$, the likelihood function can be written as

$$L(\alpha,\beta) = A(n, m-1)\prod_{t=1}^{m} f(x_t,\alpha,\beta)[1-F(x_t,\alpha,\beta)]^{R_t}; \quad x>0,\ \alpha>0,\ \beta>-1, \tag{31}$$

where

$$A(n, m-1) = n(n-R_1-1)(n-R_1-R_2-2)\ldots(n-R_1-R_2-\cdots-R_{m-1}-m+1),$$

and $f(\cdot)$ and $F(\cdot)$ are same as defined in (3) and (4), respectively. Therefore, ignoring the additive constant the log-likelihood function is written as

$$\begin{aligned}\log(L(\alpha,\beta)) &= m\log(\alpha) + m\log(1+\beta) - \alpha\sum_{t=1}^{m} x_t - \sum_{t=1}^{m}(R_t+2)\log(1+\beta e^{-\alpha x_t}) \\ &\quad + \log(1+\beta)\sum_{t=1}^{m} R_t - \alpha\sum_{t=1}^{m} x_t R_t. \end{aligned}\tag{32}$$

To compute the MLEs of the unknown parameters α and β, consider the two normal equations;

$$\frac{\partial\log(L)}{\partial\alpha} = \frac{m}{\alpha} - \sum_{t=1}^{m} x_t(1+R_t) + \frac{\beta\sum_{t=1}^{m}(R_t+2)e^{-\alpha x_t}x_t}{(1+\beta e^{-\alpha x_t})} = 0, \tag{33}$$

and

$$\frac{\partial\log(L)}{\partial\beta} = \frac{m}{1+\beta} + \frac{\sum_{t=1}^{m} R_t}{1+\beta} - \frac{\sum_{t=1}^{m}(R_t+2)e^{-\alpha x_t}}{(1+\beta e^{-\alpha x_t})} = 0, \tag{34}$$

whose solution provide the MLEs $\widehat{\alpha}$ and $\widehat{\beta}$.

Once MLEs of α and β are obtained as $\widehat{\alpha}$ and $\widehat{\beta}$, the MLEs of $R(t)$ and $h(t)$ can be derived using invariance property of MLEs as

$$\widehat{R}(t) = \frac{(1+\widehat{\beta})e^{-\widehat{\alpha}t}}{1+\widehat{\beta}e^{-\widehat{\alpha}t}}, \quad t>0 \tag{35}$$

and

$$\hat{h}(t) = \frac{\widehat{\alpha}}{1+\widehat{\beta}e^{-\widehat{\alpha}t}}, \quad t>0. \tag{36}$$

7 Simulation Study

In this Section, a simulation study is conducted to observe the behavior of the proposed method for different sample sizes, different effective sample sizes and for different censoring schemes. We have considered different sample sizes; $n = 20, 25, 30$; different effective sample sizes; $m = 8, 10, 15$ and different censoring schemes. In all the cases we have used $\alpha = 1$ and $\beta = 1$. For a given set of n, m and a censoring scheme, using the algorithm proposed by Balakrishnan and Sandhu [5], a sample is generated. Using the sample, the MLEs of unknown parameters α and β are computed based on the method proposed in Section 6.

Finally, with 1000 replications, using a program in R, the MLEs of α, β, $R(t)$ and $h(t)$ along with their mean square errors (MSEs) are obtained. The results are presented in Tables 1, 2 and 3.

Table 1: MLEs of α and β along with their MSEs for different censoring schemes, for $\alpha = 1$ and $\beta = 1$

n	m	Censoring scheme	$\widehat{\alpha}$	MSE($\widehat{\alpha}$)	$\widehat{\beta}$	MSE($\widehat{\beta}$)
20	8	(7*0,12)	1.0349	0.01610	1.0505	0.01332
20	8	(12,7*0)	1.0327	0.01534	1.0170	0.01175
20	8	(3, 0, 5, 2, 1, 1, 0, 0)	1.0286	0.01570	1.0217	0.01276
20	10	(9*0,10)	1.0219	0.01553	1.0406	0.01282
20	10	(10,9*0)	1.0285	0.01512	0.9901	0.01330
20	10	(3,0,5,2,6*0)	1.0240	0.01551	0.9839	0.01279
20	20	(20*0)	1.0093	0.01272	1.0058	0.01095
25	10	(9*0,15)	1.0184	0.01422	1.0229	0.01204
25	10	(15,9*0)	1.0212	0.01499	1.0191	0.01228
25	10	(5,5,5,7*0)	1.0466	0.01575	0.9963	0.01322
25	10	(3,0,5,2,1,1,0,0,2,1)	1.0223	0.01460	1.0076	0.01123
25	15	(14*0,10)	1.0141	0.01356	1.0188	0.01153
25	15	(10,14*0)	1.0119	0.01304	0.9878	0.01170
25	25	(25*0)	1.0067	0.00552	1.0057	0.00748
30	10	(9*0,20)	1.0122	0.01137	1.0214	0.01158
30	10	(20,9*0)	1.0107	0.01215	1.0116	0.01061
30	10	(5,0,5,0,0,5,0,0,0,5)	1.0246	0.01540	1.0374	0.01304
30	10	(3,0,5,2,1,1,2,2,2,2)	1.0265	0.01500	1.0124	0.01042
30	15	(14*0,15)	1.0103	0.01112	1.0110	0.01096
30	15	(15,14*0)	1.0102	0.01116	1.0091	0.01015
30	15	(3,0,5,2,1,1,0,0,1,0,2,4*0)	1.0202	0.01463	1.0049	0.01036
30	30	(30*0)	1.0005	0.00480	1.0020	0.00519

Table 2: MLEs of $R(t)$ and $h(t)$ along with their MSEs for different censoring schemes, for $\alpha = 1$ and $\beta = 1$; $t = 0.2$

$t = 0.2$; $R(t) = 0.9003$; $h(t) = 0.5498$

n	m	Censoring scheme	$\hat{R}(t)$	$MSE(\hat{R}(t))$	$\hat{h}(t)$	$MSE(\hat{h}(t))$
20	8	(7*0,12)	0.9027	0.0022	0.5483	0.0240
20	8	(12,7*0)	0.8947	0.0016	0.5910	0.0191
20	8	(3, 0, 5, 2, 1, 1, 0, 0)	0.8956	0.0015	0.5864	0.0177
20	10	(9*0,10)	0.9031	0.0021	0.5445	0.0207
20	10	(10,9*0)	0.8919	0.0015	0.6080	0.0171
20	10	(3,0,5,2,6*0)	0.8928	0.0016	0.6025	0.0212
20	20	(20*0)	0.9009	0.0011	0.5512	0.0102
25	10	(9*0,15)	0.9023	0.0020	0.5508	0.0202
25	10	(15,9*0)	0.8936	0.0014	0.5971	0.0132
25	10	(5,5,5,7*0)	0.8988	0.0016	0.5670	0.0202
25	10	(3,0,5,2,1,1,0,0,2,1)	0.9053	0.0018	0.5301	0.0245
25	15	(14*0,10)	0.8968	0.0017	0.5821	0.0199
25	15	(10,14*0)	0.8939	0.0013	0.5954	0.0187
25	25	(25*0)	0.9007	0.0010	0.5501	0.0101
30	10	(9*0,20)	0.9014	0.0017	0.5573	0.0167
30	10	(20,9*0)	0.8952	0.0016	0.5880	0.0147
30	10	(5,0,5,0,0,5,0,0,0,5)	0.8984	0.0018	0.5721	0.0136
30	10	(3,0,5,2,1,1,2,2,2,2)	0.8981	0.0018	0.5736	0.0177
30	15	(14*0,15)	0.9005	0.0016	0.5612	0.0160
30	15	(15,14*0)	0.8942	0.0013	0.5932	0.0156
30	15	(3,0,5,2,1,1,0,0,1,0,2,4*0)	0.8929	0.0015	0.6016	0.0143
30	30	(30*0)	0.9003	0.0008	0.5499	0.0090

Table 3: MLEs of $R(t)$ and $h(t)$ along with their MSEs for different censoring schemes, for $\alpha = 1$ and $\beta = 1$; $t = 1$

$t = 1$; $R(t) = 0.5379$; $h(t) = 0.7311$

n	m	Censoring scheme	$\hat{R}(t)$	$MSE(\hat{R}(t))$	$\hat{h}(t)$	$MSE(\hat{h}(t))$
20	8	(7*0,12)	0.5422	0.0030	0.7248	0.0133
20	8	(12,7*0)	0.5388	0.0021	0.7464	0.0136
20	8	(3, 0, 5, 2, 1, 1, 0, 0)	0.5179	0.0025	0.7554	0.0163
20	10	(9*0,10)	0.5213	0.0029	0.7639	0.0170
20	10	(10,9*0)	0.5345	0.0016	0.7480	0.0142
20	10	(3,0,5,2,6*0)	0.5371	0.0014	0.7589	0.0143
20	20	(20*0)	0.5370	0.0013	0.7354	0.0133
25	10	(9*0,15)	0.5336	0.0028	0.7438	0.0140
25	10	(15,9*0)	0.5316	0.0017	0.7541	0.0141
25	10	(5,5,5,7*0)	0.5318	0.0016	0.7660	0.0154
25	10	(3,0,5,2,1,1,0,0,2,1)	0.5267	0.0021	0.7417	0.0137
25	15	(14*0,10)	0.5387	0.0028	0.7476	0.0155
25	15	(10,14*0)	0.5215	0.0019	0.8001	0.0168
25	25	(25*0)	0.5374	0.0011	0.7322	0.0115
30	10	(9*0,20)	0.5381	0.0023	0.7360	0.0125
30	10	(20,9*0)	0.5400	0.0022	0.7565	0.0142
30	10	(5,0,5,0,0,5,0,0,0,5)	0.5242	0.0028	0.7603	0.0156
30	10	(3,0,5,2,1,1,2,2,2,2)	0.5251	0.0027	0.7481	0.0150
30	15	(14*0,15)	0.5360	0.0021	0.7432	0.0137
30	15	(15,14*0)	0.5349	0.0016	0.7340	0.0101
30	15	(3,0,5,2,1,1,0,0,1,0,2,4*0)	0.5355	0.0018	0.7464	0.0111
30	30	(30*0)	0.5378	0.0010	0.7313	0.0063

8 Conclusion

From Table 1, we observe that for complete samples, MLEs of α and β are very nearly unbiased and can be regarded as good estimators. It is also observed that for complete samples, as sample size n increases the average MSE decreases. Also, the MSE generally decreases as the failure information m increases, and for all the censoring schemes the MSE of the estimates are quite small and can be used in all practical situations. Here one has to make a trade off between the precision of the estimation method and the cost of the experiment.

Also, from Tables 2 and 3, it is observed that for the MLEs of $R(t)$ and $h(t)$, the MSE generally decreases as the failure information m increases. In addition, for the complete samples, as sample size n increases the average MSE decreases.

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References

[1] R. Aggarwala and N. Balakrishnan, Recurrence relations for single and product moments of progressive Type-II right censored order statistics from exponential and truncated exponential distributions, Ann. Inst. Statist. Math., 48(1996), 757–771.

[2] R. Aggarwala and N. Balakrishnan, Some properties of progressive censored order statistics from arbitrary and uniform distributions with applications to inference and simulation, Journal of Statistical planning and Inference, 70(1998), 35–49.

[3] H. Athar and Z. Akhter, Recurrence relations between moments of progressive type-II right censored order statistics from doubly truncated Weibull distribution, Statistical Papers, 57(2015), 419–428.

[4] N. Balakrishnan and R. Aggarwala, Progressive Censoring–Theory, Methods, and Applications, Birkhauser, Boston, 2000.

[5] N. Balakrishnan and R.A. Sandhu, A simple simulation algorithm for generating progressively Type-II censored samples, American Statistician, 49(1995), 229–230.

[6] F.M. Bass, A new product growth model for consumer durables, Management Science, 15(1969), 215–227.

[7] A.C. Cohen, Progressively censored samples in life testing, Technometrics, 5(1963), 327–329.

[8] A.C. Cohen, Progressively censored sampling in the three parameter log-normal distribution, Technometrics, 18(1976), 99–103.

[9] A.C. Cohen, Truncated and Censored Samples: Theory and Applications, Marcel Dekker, New York, 1991.

[10] A.C. Cohen and B.J. Whitten, Parameter Estimation in Reliability and Life Span Models, Marcel Dekker, New York, 1988.

[11] P. Jodŕa, On a connection between the polylogarithms and order statistics, XXI Congreso de Ecuaciones Diferenciales Y Aplicaciones XI Congreso de Matematica Aplicada (2009), 1–8.

[12] N. Meade and T. Islam, Modelling and forecasting the diffusion of Innovation- A 25-year review, International Journal of Forecasting, 22(2006), 519–545.

[13] N. Pushkarna, J. Saran and R. Tiwari, L-moments and TL-moments estimation and relationships for moments of progressive Type-II right censored order statistics from Frechet distribution, ProbStat Forum, 8(2015), 112–122.

[14] J. Saran, K. Nain and A.P. Bhattacharya, Recurrence relations for single and product moments of progressive Type-II right censored order statistics from left truncated Logistic distribution with application to inference, International Journal of Mathematics and Statistics, 19(2018), 113–136.

[15] J. Saran and V. Pande, Recurrence relations for moments of progressive Type-II right censored order statistics from Half-Logistic distribution, Journal of Statistical Theory and Applications, 11(2012), 87–96.

[16] J. Saran and N. Pushkarna, Recurrence relations for moments of progressive Type-II right censored order statistics from Burr distribution, Statistics, 35(4) (2001), 495–507.

[17] J. Saran and N. Pushkarna, Moments of Progressive Type-II Right Censored Order Statistics from a General Class of Doubly Truncated Continuous Distributions, Journal of Statistical Theory and Applications, 13(2014), 162–174.

[18] B. Singh and R.U. Khan, Moments of progressively type-II right censored order statistics from additive Weibull distribution, ProbStat Forum, 12(2019), 36–46.

[19] D.R. Thomas and W.M. Wilson, Linear order statistics estimation for the two parameter Weibull and extreme value distributions from type-II progressively censored samples, Technometrics, 14(1972), 679–691.