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On The Existence Of Solutions Of Noncommutative Stochastic Integral Inclusions*

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Abstract

The aim of this work is to establish the existence of solution to a non commutative stochastic integral inclusion. By using matrix elements of quantum stochastic calculus of Hudson-Parthasarathy type, an Aumann quantum stochastic integral was formulated. As an application, the existence of solution to quantum stochastic control problem was established via a noisy Ricatti differential inclusion.

1 Introduction

Quantum stochastic differential equations (qsde) of Hudson-Parthasarathy quantum stochastic calculus had undergone various reformulations [8], [6], [3]. These equations have applications in quantum optics, open quantum systems, quantum measurements, etc. Likewise quantum stochastic control and quantum filtering problems had attracted the interest of authors [7]. But of greater interest to us is the formulation of operator-valued stochastic control in Fock space with applications to orbit-tracking problems as discussed in [4]. The optimal control problem for the non commutative stochastic differential equations was established and the work had since been extended. In another development, differential inclusions have applications to classical control theory and gave a wider applications of set-valued analysis to optimal control problems [9]. In classical differential inclusions, Aumann integral [2] played a vital role in the formulation of integral inclusions for measurable set-valued maps, see [1], [5].

The aim of this work is to establish non commutative stochastic integral inclusions via set-valued approach. The existence of solution to quantum stochastic differential inclusion established has application to quantum stochastic Ricatti inclusion of quantum stochastic control theory in [4].

2 Preliminaries

In this section we shall state definitions and some preliminary results on quantum stochastic calculus (QSC) and differential inclusions which will be employed in subsequent sections.

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2.1 Notations and Definitions

Let $\mathcal{R}$ be complex separable Hilbert space called initial Hilbert space. This space describes events and observables concerning a system. In the sequel, we let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = [0, +\infty)$. Let $H = L^2(\mathbb{R}, \mathbb{C})$, the symmetric (boson) Fock space describing events and observables concerning a noise process is $\Gamma(H)$ (later denoted by Γ). A natural dense subset of Γ consists of linear space generated by the set of exponential vectors in Γ of the form

$$\psi(f) = \bigoplus_{n=0}^{\infty} (n!)^{-\frac{1}{2}} \bigotimes^{n} f, \quad f \in H$$

where $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$. For $f, g \in H$, the relation is defined with respect to the inner products

$$\langle \psi(f), \psi(g) \rangle_\Gamma = \exp \langle f, g \rangle_H,$$

which is assumed to be antilinear in the first component and linear in the second component.

Events and observables concerning a system plus noise was described by $\mathcal{R} \otimes \Gamma$. Let $\mathbb{D}$ be a linear dense subspace of $\mathcal{R}$ and $\mathbb{E}$ a linear space generated by the set of exponential vectors dense in Γ. The linear span of a linearly independent set $\{c \underline{\otimes} \psi(f) : c \in \mathbb{D}, f \in H\}$ is dense in $\mathcal{R} \otimes \Gamma$, where $\underline{\otimes}$ denotes algebraic tensor product. The inner product and norm induced by $\mathcal{R} \otimes \Gamma$ are respectively represented by $\langle ., . \rangle$ and $\|.\|$.

We shall introduce the notions of time and adaptedness as defined in the literatures. Let ξ be a spectral measure on $(\mathbb{R}_+, \beta)$ (where β is the σ-algebra of Borel measurable subsets of $[0, +\infty)$ whose values are projection operators on H such that $\xi(\mathbb{R}_+) = Id$, the identity operator on H. Time is defined as a $\mathbb{R}_+$-valued observable ξ with no jump points i.e. $\xi(\{t\}) = 0$ for every $t \geq 0$.

Let $I = [0, +\infty)$, a stochastic process indexed by I is an $\mathcal{R} \otimes \Gamma$-valued map, $X : I \to \mathcal{R} \otimes \Gamma$. Let $\mathbb{D} \subset \mathcal{R}$ and $\Lambda \subset H$ be linear manifolds such that for all $0 \leq s < t < +\infty$, $\xi([s, t])f \in \Lambda$ whenever $f \in \Lambda$. Also, let $\mathbb{D}_0$ and $\mathbb{E}_0$ be the linear spans of $\mathbb{D}$ and $\Gamma(\Lambda)$ respectively, then the linear span of $\mathbb{D} \otimes \Gamma(\Lambda)$ is $\mathbb{D}_0 \underline{\otimes} \mathbb{E}_0$. For each $t \geq 0$, let $H_{t]}$ denotes the range of $\xi([0, t])$; $f_{t]}$ and $f_{[t}$ respectively denote $\xi([0, t])f$ and $\xi([t, +\infty))f$, the notion of adaptedness is defined as follows:

A family $X = \{X(t) : t \in I\}$ of operators from $\mathcal{R} \otimes \Gamma$ to $\mathcal{R} \otimes \Gamma$ is called an adapted stochastic process with respect to the triple $(\xi, \mathbb{D}, \Lambda)$ if for all $t \in I$, $c \in \mathbb{D}$ and $f \in \Lambda$,

(i) $dom(X(t)) \supset \mathbb{D}_0 \underline{\otimes} \mathbb{E}_0$,

(ii) $X(t)c \otimes \psi(f_{t]}) \in \mathcal{R} \otimes \Gamma(H_{t]})$,

(iii) $X(t)c \otimes \psi(f) = (X(t)c \otimes \psi(f_{t]})) \otimes \psi(f_{[t})$.

An adapted stochastic process X is said to be continuous if for every $c \in \mathbb{D}$ and $f \in \Lambda$ the map $t \in [0, \infty) \to X(t)c \otimes \psi(f)$ is continuous. Let $\eta = c \otimes \psi(f) \in \mathcal{R} \otimes \Gamma$, an adapted stochastic process X is said to be bounded if

$$\|X(t)\eta\| = \langle X(t)\eta, X(t)\eta \rangle^{\frac{1}{2}} < \infty \text{ for every } c \in \mathbb{D},\ f \in \Lambda \text{ and } t \in I.$$

Let $B(\mathcal{R}\otimes\Gamma)$ be the space of bounded adapted stochastic processes on $\mathcal{R}\otimes\Gamma$. The norm $\| \,.\, \|'$ is defined on $B(\mathcal{R}\otimes\Gamma)$ as follows

$$\|X\|' = \sup\{\|X(t)\eta\| : \eta \in \mathcal{R}\otimes\Gamma,\ t \in I\}.$$

$B(\mathcal{R}\otimes\Gamma)$ with the norm $\| \,.\, \|'$ is a Banach space which will be denoted in the sequel by $\mathcal{B}$. Let Cl($\mathcal{B}$)(resp. Comp($\mathcal{B}$)) denote the family of all nonempty closed bounded (resp. compact) subsets of $\mathcal{B}$. For $X \in \mathcal{B}$, $\mathcal{M},\mathcal{N} \in$ Cl($\mathcal{B}$), define

$$\rho(\mathcal{M},\mathcal{N}) \equiv \max(\delta(\mathcal{M},\mathcal{N}),\delta(\mathcal{N},\mathcal{M})),$$

where

$$\delta\left(\mathcal{M},\mathcal{N}\right) \equiv \sup_{X\in\mathcal{M}} \mathbf{d}(X,\mathcal{N}) \ \text{ and } \ \mathbf{d}(X,\mathcal{N}) \equiv \inf_{Y\in\mathcal{N}} \|X-Y\|.$$

Moreover, if $\mathcal{M} \in Cl(\mathcal{B})$, then $\|\mathcal{M}\|_{cl(\mathcal{B})}$ is defined by

$$\|\mathcal{M}\|_{cl(\mathcal{B})} \equiv \rho(\mathcal{M},\{0\}).$$

The function ρ: Cl($\mathcal{B}$)$\times$ Cl($\mathcal{B}$)$\rightarrow \mathbb{R}_+$ is a metric on Cl($\mathcal{B}$) called the Hausdorff metric in Cl($\mathcal{B}$). The Hausdorff topology induced by the metric is derived as follows:

Given $\epsilon > 0$ and $\mathcal{A} \in$ Cl($\mathcal{B}$), we define an open neighbourhood $U(\mathcal{A},\epsilon)$ as

$$U(\mathcal{A},\epsilon) = \{X \in \mathcal{A} : \mathbf{d}(X,\mathcal{A}) \le \epsilon\}.$$

For every $\mathcal{M},\mathcal{N} \in$ Cl($\mathcal{B}$), the Hausdorff topology, τ_H is derived from Hausdorff metric ρ as

$$\rho(\mathcal{M},\mathcal{N}) = \inf\{\epsilon > 0 : \mathcal{M} \subset U(\mathcal{N},\epsilon) \text{ and } \mathcal{N} \subset U(\mathcal{M},\epsilon)\}.$$

THEOREM 1. The metric space (Cl($\mathcal{B}$), ρ) is complete.

PROOF. We shall prove that for any Cauchy sequence $(\mathcal{A}_n)$ of Cl($\mathcal{B}$), $\mathcal{A}_n$ converges to $\mathcal{A}$ where

$$\mathcal{A} = \bigcap_{n=1}^{\infty} \overline{\bigcup_{m=n}^{\infty}} \mathcal{A}_m \neq \emptyset.$$

For any $\epsilon > 0$ and each $k \in \mathbb{N}$ there exists N_k such that $n,m \ge N_k$ implies $\rho(\mathcal{A}_n,\mathcal{A}_m) < 2^{-k}\epsilon$. Let (n_k) be a strictly increasing sequence of $\mathbb{N}$ such that $n_k \ge N_k$. Let $x_0, x_1, x_2, ..., x_k$ be chosen such that

$$x_i \in \mathcal{A}_{n_i}, \quad \| x_{i+1} - x_i \| < 2^{-i}\epsilon, \text{ for } i = 1,2,...,k-1.$$

Now, since $\mathbf{d}(x_k,\mathcal{A}_{n_{k+1}}) \le \rho(\mathcal{A}_{n_k},\mathcal{A}_{n_{k+1}}) < 2^{-(k+1)}\epsilon$, we can choose x_{k+1} in $\mathcal{A}_{n_{k+1}}$ which satisfies $\| x_{k+1} - x_k \| < 2^{-(k+1)}\epsilon$. Therefore (x_k) is a Cauchy sequence of $\mathcal{B}$, since $\mathcal{B}$ is complete, there is $x \in \mathcal{B}$ such that $\|x_k - x\| \to 0$ as $k \to \infty$. We have $x \in \mathcal{A}$ and $\|x - x_0\| \le 2\epsilon$. Then for every $n_0 \ge N_0$ and $x_0 \in \mathcal{A}_{n_0}$ there exists a point $x \in \mathcal{A}$ such that $\|x - x_0\| \le 2\epsilon$. Hence $\delta(\mathcal{A}_{n_0},\mathcal{A}) \le 2\epsilon$ for $n_0 \ge N_0$. Let $N \in \mathbb{N}$ be such that

$m, n \geq N$ implies that $\rho(\mathcal{A}_n, \mathcal{A}_m) \leq \epsilon$. Let $x \in \mathcal{A}$, then $x \in \overline{\bigcup_{m=n}^{\infty} \mathcal{A}_m}$ and there exists $n_0 \geq N$, $y \in \mathcal{A}_{n_0}$ such that $\| x - y \| \leq \epsilon$. For each $m \geq N$, we have

$$\mathbf{d}(x, \mathcal{A}_m) \leq \mathbf{d}(x, \mathcal{A}_{n_0}) + \delta(\mathcal{A}_{n_0}, \mathcal{A}) \leq 2\epsilon.$$

Hence $\delta(\mathcal{A}, \mathcal{A}_m) \leq 2\epsilon$, which implies that $\delta(\mathcal{A}, \mathcal{A}_n) \to 0$ as $n \to \infty$. This together with $\delta(\mathcal{A}_n, \mathcal{A}) \to 0$ as $n \to \infty$ from above imply that $\rho(\mathcal{A}_n, \mathcal{A}) \to 0$ as $n \to \infty$, hence the proof.

In the sequel, we shall denote the topological space (Cl($\mathcal{B}$), τ_H) by Cl($\mathcal{B}$) and the set of all adapted stochastic processes on $\mathcal{B}$ shall be denoted by Ad($\mathcal{B}$).

2.2 Quantum Stochastic Integral

DEFINITION. A member X of Ad($\mathcal{B}$) is called

(i) absolutely continuous if the map $t \mapsto \|X(t)\|$, $t \in I$ is absolutely continuous.

(ii) locally absolutely p-integrable if $\|X(.)\|^p$ is Lebesgue -measurable and integrable on $[0, t) \subseteq I$ for each $t \in I$. We denote by $\text{Ad}(\mathcal{B})_{ac}$ (resp.$L^p_{loc}(\mathcal{B})$) the set of all absolutely continuous(resp. locally absolutely p-integrable) members of Ad($\mathcal{B}$).

Stochastic integrators

Let $L^{\infty}_{loc}(\mathbb{R}_+, \mathbb{C})$ [resp.$L^{\infty}_{\mathcal{B},loc}(\mathbb{R}_+)$] be the linear space of all measurable, locally bounded functions from $\mathbb{R}_+$ to $\mathbb{C}$ [resp. to $\mathcal{B}$].

For $f, g \in H$ and $\pi \in L^{\infty}_{\mathcal{B},loc}(\mathbb{R}_+)$, the annihilation, creation and gauge operators are respectively linear operators $a(f)$, $a^+(f)$, $\lambda(\pi) : \Gamma \to \Gamma$ defined as:

$$\begin{aligned}
a(f)\psi(g) &= \langle f, g \rangle_H \psi(g), \\
a^+(f)\psi(g) &= \frac{d}{d\sigma}\psi(g + \sigma f) \mid_{\sigma=0}, \\
\lambda(\pi)\psi(g) &= \frac{d}{d\sigma}\psi(e^{\sigma\pi} f) \mid_{\sigma=0} .
\end{aligned}$$

They give rise to the operator-valued maps A_f, A_f^+ and Λ_π defined by

$$\begin{aligned}
A_f(t) &\equiv a(f\chi_{[0,t)}), \\
A_f^+(t) &\equiv a^+(f\chi_{[0,t)}), \\
\Lambda_\pi(t) &\equiv \lambda(\pi\chi_{[0,t)}),
\end{aligned}$$

for $t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A_f^+ and Λ_π are stochastic processes , called annihilation, creation and gauge processes , respectively, when their values are identified with their ampliations on $\mathcal{R} \otimes \Gamma$. These are the stochastic integrators in Hudson and Parthasarathy[10] formulation of boson quantum stochastic integration. For processes $p, q, u, v \in L^2_{loc}(\mathcal{B})$, the quantum stochastic integral:

$$\int_{t_0}^{t} \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds\big), \quad t_0, t \in \mathbb{R}_+,$$

is interpreted in the sense of Hudson-Parthasarathy[10].

DEFINTION.

(a) By a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$ we mean a multifunction on I with values in $\mathrm{Cl}(\mathcal{B})$; that is, $\Phi : I \to 2^{\mathcal{B}}$, such that $\Phi(t) \in Cl(\mathcal{B})$.

(b) If Φ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of Φ is a stochastic process $X : I \to \mathcal{B}$ with the property that $X(t) \in \Phi(t)$ for almost all $t \in I$.

(c) A multivalued stochastic process Φ will be called (i) measurable if $t \mapsto \mathbf{d}(x, \Phi(t))$ is measurable for arbitrary $x \in \mathcal{B}$. (ii) locally absolutely p-integrable if $t \mapsto \| \Phi(t) \|_{cl\mathcal{B}}$, $t \in \mathbb{R}_+$ lie in $L^p_{loc}(I)$. For $p \in (0, \infty)$ and $I \subseteq \mathbb{R}_+$, the set of all locally absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\mathcal{B})_{mvs}$. Moreover, if $\Phi \in L^p_{loc}(\mathcal{B})_{mvs}$, then we denote by

$$L_p(\Phi) \equiv \{\phi \in L^p(\mathcal{B}) : \phi \text{ is a selection of } \Phi\}.$$

Let $f, g \in H$, $\pi \in L^\infty_{\mathcal{B},loc}(\mathbb{R}_+)$,and let the identity map on $\mathcal{R} \otimes \Gamma$ be denoted by $\mathbb{I}$, if N is any of the stochastic processes A_f, A^+_g, Λ_π and $s \mapsto s\mathbb{I}$, $s \in \mathbb{R}_+$. We introduce the *Aumann quantum stochastic integral* as follows: If $\Phi \in L^2_{loc}(\mathcal{B})_{mvs}$, then

$$\int_{t_0}^{t} \Phi(s)dN(s) \equiv \left\{ \int_{t_0}^{t} \phi(s)dN(s) : \phi \in L_2(\Phi) \right\}$$

where $\int_{t_0}^{t} \phi(s)dN(s) : \mathcal{H} \otimes \Gamma \to \mathcal{H} \otimes \Gamma$ is a linear operator defined on the linear span $\{c \otimes \psi(f) : c \in \mathbb{D}, f \in \Lambda\}$ with matrix elements

$$\langle c \otimes \psi(f), \int_{t_0}^{t} \phi(s)dN(s)d \otimes \psi(g)\rangle.$$

This leads to the following definition of quantum stochastic integral inclusion in the sense of Aumann: Let $E, F, G, H \in L^2_{loc}(\mathcal{B})_{mvs}$ with selections $p, q, u, v \in L^2_{loc}(\mathcal{B})$ respectively. For $f, g \in L^\infty_{loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, let the integral

$$\begin{aligned} M(t) &= \int_{t_0}^{t} \big(E(s)d\Lambda_\pi(s) + F(s)dA_f(s) \\ &\quad + G(s)dA^+_g(s) + H(s)ds\big), \quad a.e. t \in I, \end{aligned}$$

then

$$M(t) = \Big\{ \int_{t_0}^{t} \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds\big) \\ : p(s) \in E(s), q(s) \in F(s), u(s) \in G(s), v(s) \in H(s) \Big\}.$$

Therefore for a fixed $x(t_0) = x_0$, we have the quantum stochastic integral inclusion

$$x(t) \in x_0 + \int_{t_0}^{t} \big(E(s)d\Lambda_\pi(s) + F(s)dA_f(s) \\ + G(s)dA_g^+(s) + H(s)ds\big), \quad a.e. t \in I.$$

Let $N \subset \mathcal{B}$, a multivalued map $\Phi : I \times \mathcal{N} \to 2^{\mathcal{B}}$ will be said to be upper semicontinuous at a point $(t_0, x_0) \in I \times \mathcal{N}$, if every $\epsilon > 0$, there exists $\delta = \delta((t_0, x_0), \epsilon) > 0$ such that

$$\Phi(t, x) \subset \Phi(t_0, x_0) + \epsilon B,$$

where B is a unit ball centred at the origin. Φ is said to be upper semicontinuous on $I \times \mathcal{N}$ if it is upper semicontinuous at every point $(t, x) \in I \times \mathcal{N}$.

3 Existence Results

In this subsection we prove an existence theorem for upper semicontinuous quantum stochastic differential inclusions.
The following proposition follows from Theorem 4.1 in [10].

PROPOSITION 1. Let $E, F, G, H \in L^2_{loc}(I \times \mathcal{B})_{mvs}$ with selections $p, q, u, v \in L^2(\mathcal{B})$ respectively. For arbitrary $\eta, \xi \in \mathbb{D} \otimes \mathbb{E}$ with $\eta = c \otimes e(\alpha)$, $\xi = d \otimes e(\beta)$, $\alpha, \beta \in L^2_{loc}(\mathbb{R}_+)$, $f, g \in L^\infty_{loc}(\mathbb{R}_+)$, $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, then

$$\langle \eta, M(t)\xi \rangle = \Big\{ \langle \eta, \int_{t_0}^{t} \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA_g^+(s) + v(s)ds\big)\xi \rangle \\ : p(s) \in E(s, x(s)), q(s) \in F(s, x(s)), u(s) \in G(s, x(s)), v(s) \in H(s, x(s)) \Big\}.$$

The proposition therefore gives the definition for the matrix form of the Aumann integral for multifunction, that is, for each $t \in I$; $M(t) : \mathcal{H} \otimes \Gamma \to 2^{\mathcal{H} \otimes \Gamma}$. We denote by $\mathcal{A}(\mathcal{H} \otimes \Gamma)$, the set of matrix form Aumann integral maps on $\mathcal{H} \otimes \Gamma$. We can define a metric d on $\mathcal{A}(\mathcal{H} \otimes \Gamma)$ as follows: for each $t \in I$, $\eta, \xi \in \mathcal{H} \otimes \Gamma$, let $\Psi_1(t) \equiv \langle \eta, M_1(t)\xi \rangle$ and $\Psi_2(t) \equiv \langle \eta, M_2(t)\xi \rangle \in \mathcal{A}(\mathcal{H} \otimes \Gamma)$ such that

$$M_1(t) = \int_{t_0}^{t} \big(E_1(s, x(s))d\Lambda_\pi(s) + F_1(s, x(s))dA_f(s) \\ + G_1(s, x(s))dA_g^+(s) + H_1(s, x(s))ds\big)$$

and

$$M_2(t) = \int_{t_0}^{t} \big(E_2(s,x(s))d\Lambda_\pi(s) + F_2(s,x(s))dA_f(s) + G_2(s,x(s))dA_g^+(s) + H_2(s,x(s))ds\big)$$

for some $E_i, F_i, G_i, H_i \in L^2_{loc}(I \times \mathcal{B})_{mvs}$ with selections $p_i, q_i, u_i, v_i \in L^2_{loc}(\mathcal{B})$ respectively, $i = 1, 2$,

$$d(\Psi_1(t), \Psi_2(t)) = \int_{t_0}^{t} \max\{\rho(E_1, E_2), \rho(F_1, F_2)\rho(G_1, G_2)\rho(H_1, H_2)\}ds.$$

We remark that $(\mathcal{A}(\mathcal{H}\otimes\Gamma), d)$ is a complete metric space. We shall prove the existence of solutions to a quantum stochastic integral inclusions in form of matrix elements.

LEMMA1. Suppose A is a closed convex set in $cl(\mathcal{B})$ then

$$\langle \eta, (\int_{t_1}^{t_2} Ads)\xi\rangle = \langle \eta, (t_2 - t_1)A\xi\rangle.$$

PROOF. For arbitrary $\eta, \xi \in \mathcal{H} \otimes \Gamma$,

$$\begin{aligned} \langle \eta, (t_2 - t_1)A\xi\rangle &= \Big\{\langle \eta, (t_2 - t_1)p\xi\rangle : p \in A\Big\} \\ &\subset \Big\{\langle \eta, (\int_{t_1}^{t_2} p\xi)ds : p \in A\Big\} \\ &= \Big\{\int_{t_1}^{t_2} \langle \eta, p\xi\rangle ds : p \in A\Big\} \\ &= \Big\{\langle \eta, \int_{t_1}^{t_2} p\xi\rangle ds : p \in A\Big\} \\ &= \langle \eta, (\int_{t_1}^{t_2} Ads)\xi\rangle. \end{aligned}$$

Now let $z \in \langle \eta, (\int_{t_1}^{t_2} Ads)\xi\rangle$, this implies that $z = \langle \eta, (\int_{t_1}^{t_2} p(s)ds)\xi\rangle$ where $p(.)$ is measurable with values in A. By Mean value Theorem, $z = \langle \eta, (t_2 - t_1)p\xi\rangle$, $p \in \overline{co}\{p(s) : t_1 \le s \le t_2\}$ which implies that $z \in \langle \eta, (t_2 - t_1)A\xi\rangle$. Hence this lemma holds.

THEOREM 2. Let $E, F, G, H \in L^2_{loc}(I \times \mathcal{B})_{mvs}$ be upper semicontinuous multivalued stochastic processes from $I \times \mathcal{B}$ into the compact convex subsets of $\mathcal{B}$. Then $x(.) \in Ad(\mathcal{B})$ is a solution on I to the differential inclusion

$$\langle \eta, x'(t)\xi\rangle \in \langle \eta, M'(t)\xi\rangle \tag{1}$$

if and only if

$$\langle \eta, (x(t_2) - x(t_1))\xi\rangle \in \langle \eta, M(t)\xi\rangle. \tag{2}$$

PROOF. When $x(.) \in Ad(\mathcal{B})$ is a solution to (1) on I, its derivative is a measurable selection of

$$\big(E(s,x(s))d\Lambda_\pi(s) + F(s,x(s))dA_f(s) + G(s,x(s))dA_g^+(s) + H(s,x(s))ds\big).$$

Hence,

$$\langle \eta, (x(t_2) - x(t_1))\xi \rangle \in \langle \eta, M(t)\xi \rangle.$$

Conversely, assume that (2) holds and let $\|\Phi\| = 4\Phi^m$ where $\Phi^m = \max\{\|E\|, \|F\|, \|G\|, \|H\|\}$. Then

$$\begin{aligned} \|x(t_2) - x(t_1)\|_\gamma &\leq \int_{t_1}^{t_2} \big\| \big(E(s,x(s))d\Lambda_\pi(s) + F(s,x(s))dA_f(s) \\ &\quad + G(s,x(s))dA_g^+(s) + H(s,x(s))ds\big) \big\| \\ &\leq \Big(\|E\| + \|F\| + \|G\| + \|H\| \Big) |t_2 - t_1| \\ &\leq \|\Phi\| \, |t_2 - t_1| \,. \end{aligned}$$

This implies that $x(.)$ is Lipschitzian and hence differentiable a.e. $t \in I$. Let t' be a point where $x'(t)$ exists. Since E, F, G, H are upper semicontinuous, fix $\epsilon > 0$, let B be a unit ball in $\mathcal{B}$ and let $\delta > 0$ be such that $|t - t'| \leq \delta$ implies

$$\begin{aligned} &\Big[E(t', x(t')) + F(t', x(t')) + G(t', x(t')) + H(t', x(t'))\Big] \\ \subset\ &\Big[E(t, x(t)) + F(t, x(t)) + G(t, x(t)) + H(t, x(t)) + \epsilon B\Big]. \end{aligned}$$

Then

$$\begin{aligned} \langle \eta, (x(t_1) - x(t))\xi \rangle &\in \langle \eta, M(t)\xi \rangle \\ &\subset \langle \eta, M(t)\xi \rangle + \langle \eta, \epsilon p \xi \rangle \qquad (p \in B) \\ &= \langle \eta, (t_1 - t)M'(t)\xi \rangle + \langle \eta, \epsilon p \xi \rangle, \end{aligned}$$

which implies that $\langle \eta, x'(t)\xi \rangle \in \langle \eta, M'(t)\xi \rangle + \langle \eta, \epsilon p \xi \rangle$. Since ϵ is arbitrarily chosen and E, F, G, H closed, we see that

$$\langle \eta, x'(t)\xi \rangle \in \langle \eta, M'(t)\xi \rangle.$$

3.1 Application to Quantum Stochastic Control

Consider the quantum stochastic Ricatti differential inclusion

$$\begin{cases} dP(t) \in (P(t)\Omega(t) + \Omega^*(t)P(t) + \Phi^*(t)P(t)\Phi(t) - P^2(t) + Q(t))dt \\ \qquad\quad + (P(t)\Psi(t) + \Phi^*(t)P(t))dA_f(t) + (P(t)\Phi(t) + \Psi^*(t)P(t))dA_g^+(t), \\ P(0) = P_0 \end{cases} \tag{3}$$

Where $\Omega(t), \Phi(t), \Psi(t), Q(t) \in L^2_{loc}(\mathcal{B})_{mvs}$, $t \in [0,T]$, and $\Omega^*, \Phi^*, \Psi^*, Q^* \in L^2_{loc}(\mathcal{B})_{mvs}$ are their adjoints respectively. The matrix elements integral equivalent of (3.3) is

$$\begin{aligned} & \langle \eta, P(t)\xi \rangle \\ \in \ & \langle \eta, P_0\xi \rangle + \langle \eta, [\int_t^T (P(s)\Omega(s) + \Omega^*(s)P(s) + \Phi^*(s)P(s)\Phi(s) - P^2(s) + Q(s))ds \\ & +(P(s)\Psi(s) + \Phi^*(s)P(s))dA_f(s) + (P(s)\Phi(s) + \Psi^*(s)P(s))dA_g^+(s)]\xi \rangle. \end{aligned} \tag{4}$$

Suppose $\Omega(t,P), \Phi(t,P), \Psi(t,P)$ and their adjoints are upper semicontinuous on $I \times \mathcal{B}$ and $Q(t)$ is upper semicontinuous on $\mathcal{B}$. Let $\Omega(t,P) \equiv P(t)\Omega(t)$, $\Phi(t,P) \equiv P(t)\Phi(t)$ and $\Psi(t,P) \equiv P(t)\Psi(t)$ where $\Omega(t)$, $\Phi(t)$, $\Psi(t)$ and their adjoints are upper semicontinuous on $\mathcal{B}$, such that $\Omega(t,P)^* \equiv \Omega(t)^*P(t)^*$, $\Phi(t,P)^* \equiv \Phi(t)^*P(t)^*$ and $\Psi(t,P)^* \equiv \Psi(t)^*P(t)^*$.

The following result is a corollary to Theorem 1 above and establishes the existence of solution to (3) or (4).

COROLLARY 1. Assume that the maps Ω, Φ, $\Psi \in L^2_{loc}(I \times \mathcal{B})_{mvs}$ and $P, Q \in L^2_{loc}(\mathcal{B})$ with compact convex values such that

(a) $t \mapsto \Omega(t, P(t))$, $t \mapsto \Phi(t, P(t))$, $t \mapsto \Psi(t, P(t))$ have measurable selections,

(b) $P \mapsto \Omega(t, P(t)), P \mapsto \Phi(t, P(t)), P \mapsto \Psi(t, P(t))$ and their adjoints are upper semicontinuous.

(c) For any $t_1, t_2 \in [0,T]$,

$$\begin{aligned} & \langle \eta, (P(t_2) - P(t_1))\xi \rangle \\ \in \ & \langle \eta, [\int_{t_1}^{t_2} (P(s)\Omega(s) + \Omega^*(s)P(s) + \Phi^*(s)P(s)\Phi(s) - P^2(s) + Q(s))ds \\ & +(P(s)\Psi(s) + \Phi^*(s)P(s))dA_f(s) + (P(s)\Phi(s) + \Psi^*(s)P(s))dA_g^+(s)]\xi \rangle. \end{aligned}$$

Then the quantum stochastic Ricatti differential inclusion (3) has a solution on $[0,T]$.

PROOF. Suppose ω, ϕ, ψ are measurable selections of Ω, Φ, Ψ respectively, then (c) implies that

$$\begin{aligned} & \langle \eta, (P(t_2) - P(t_1))\xi \rangle \\ = \ & \langle \eta, [\int_{t_1}^{t_2} (P(s)\omega(s) + \omega^*(s)P(s) + \phi^*(s)P(s)\phi(s) - P^2(s) + Q(s))ds \\ & +(P(s)\psi(s) + \phi^*(s)P(s))dA_f(s) + (P(s)\Phi(s) + \psi^*(s)P(s))dA_g^+(s)]\xi \rangle. \end{aligned}$$

From Theorem 1, there exists $K > 0$ such that

$$|\langle \eta, (P(t_2) - P(t_1))\xi \rangle| < K\,|t_2 - t_1|\,.$$

That is, P is Lipschitzian. Also, $\omega,\ \phi,\ \psi$ are bounded from the upper semicontinuity of $\Omega,\ \Phi,\ \Psi$. Therefore the quantum stochastic integral equation arising from (4) by using the measurable selection is a form of matrix element equivalence of noisy Ricatti differential equation in [4]. The existence of solution then follows as established in Lemma 2.1 in [4]. Since these measurable selections are not unique, then the solution to (3) is a set.

Suppose the solution set is the set of stochastic processes $\mathbb{P}(t) = \{P(t) : t \in [0, T]\}$ and let

$$\begin{aligned} J_{\xi,T}(u) &= \int_0^T \big[\langle X(t)\xi, R^*RX(t)\xi\rangle + \langle u(t)\xi, u(t)\xi\rangle\big]dt \\ &\quad +\langle X(T)\xi, P(T)X(T)\xi\rangle \quad \xi \in \mathcal{H}\otimes\Gamma, \end{aligned} \tag{5}$$

be the quadratic performance functional corresponding to quantum stochastic control problem

$$dX(t) = (\Omega(t)X(t) + u(t))dt + \Psi(t)X(t)dA_f(t) + \Phi(t)X(t)dA_g^+(t) \quad X(0) = I.$$

Where $u(t)$ is a continuous selection from the space of admissible controls $U(t) = \{u(t) : t \in [0, T]\}$ and R is a bounded operator on the system space $\mathcal{H}$.

Then the quadratic performance functional (5) is minimized by the feedback control

$$u(t) = -P(t)X(t).$$

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