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On r-Dynamic Coloring Of The Total Graphs Of Gear Graphs*

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Abstract

An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq \min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the r-dynamic chromatic number of the total graph of a gear graph.

1 Introduction

Graphs in this paper are simple and finite. For undefined terminologies and notations see [5, 17]. Thus for a graph G, $\delta(G)$, $\Delta(G)$ and $\chi(G)$ denote the minimum degree, maximum degree and chromatic number of G respectively. When the context is clear we write, δ, Δ and χ for brevity. For $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v in G and $d(v) = |N(v)|$. The r-dynamic chromatic number was first introduced by Montgomery [14].

An r-dynamic coloring of a graph G is a map c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$ and (ii) for each vertex $v \in V(G), |c(N(v))| \geq \min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v and $d(v)$ its degree and r is a positive integer.

The first condition characterizes proper colorings, the adjacency condition and second condition is double-adjacency condition. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number denoted by $\chi_d(G)$ [1, 2, 3, 4, 8]. By simple observation, we can show that $\chi_r(G) \leq \chi_{r+1}(G)$, however $\chi_{r+1}(G) - \chi_r(G)$ can be arbitrarily large, for example $\chi(Petersen) = 2$, $\chi_d(Petersen) = 3$, but $\chi_3(Petersen) = 10$. Thus, finding an exact values of $\chi_r(G)$ is not trivially easy.

There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example, for a graph G with $\Delta(G) \geq 3$, Lai et al. [8] proved that

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$\chi_d(G) \leq \Delta(G)+1$. An upper bound for the dynamic chromatic number of a d-regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in [7]. In fact, it was proved that $\chi_d(G) \leq \chi(G)+2\log_2 \alpha(G)+3$. Taherkhani gave in [15] an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree Δ and the minimum degree δ. i.e., $\chi_2(G)-\chi(G) \leq \lceil (\Delta e)/\delta \log \left(2e\left(\Delta^2+1\right)\right)\rceil$.

Li et al. proved in [10] that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, Li and Zhou [9] showed that to determine whether there exists a 3-dynamic coloring, for a claw free graph with the maximum degree 3, is NP-complete.

N. Mohanapriya et al. [11, 12] studied the dynamic chromatic number for various graph families. Also, it was proven in [13] that the r-dynamic chromatic number of line graph of a helm graph H_n is

$$\chi_r(L(H_n)) = \begin{cases} n-1, & \delta \leq r \leq n-2, \\ n+1, & r=n-1, \\ n+2, & r=n \text{ and } n \equiv 1 \mod 3, \\ n+3, & r=n \text{ and } n \not\equiv 1 \mod 3, \\ n+4, & r=n+1=\Delta, n\geq 6 \text{ and } 2n-2\equiv 0 \mod 5, \\ n+5, & r=n+1=\Delta, n\geq 6 \text{ and } 2n-2\not\equiv 0 \mod 5. \end{cases}$$

In this paper, we study $\chi_r(G)$, when $1 \leq r \leq \Delta$. We find the r-dynamic chromatic number of the total graph of a gear graph.

2 Preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [6, 16] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G)\cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The gear graph is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle. The gear graph G_n has $2n+1$ nodes and $3n$ edges. Let

$$V(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\},$$

$$E(G_n) = \{vv_i : 1 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n\}.$$

Also, let the vertices which in the total graph represent an edge be, $e_i = vv_i, 1 \leq i \leq n$ and $s_i = u_iv_i, s_{i+1} = u_iv_{i+1}$, $1 \leq i \leq n$.

THEOREM 1. Let $n \geq 10$, $T(G_n)$ be the total graph of a Gear graph G_n and let

$\Delta = \Delta(T(G_n))$. Then

$$\chi_r(T(G_n)) = \begin{cases} n+1, & 1 \le r \le n, \\ n+3, & \text{if } n \text{ is even}, \quad r = n+1, \\ n+4, & \text{if } n \ is \text{ odd}, \quad r = n+1, \\ n+5, & \text{if } r = n+2, \\ n+6, & r = n+3 \text{ and } n \equiv 0 \mod 6, \\ n+7, & r = n+3 \text{ and } n \not\equiv 0 \mod 6, \\ n+5, & r = n+4 \text{ and } n \equiv 0 \mod 4, \\ n+6, & r = n+4 \text{ and } n \equiv 3 \mod 4, \\ n+7, & r = n+4 \text{ and } n \not\equiv 3 \mod 4, \\ n+6, & r = n+5, \ n \equiv 2 \mod 4 \text{ and } n \equiv 3 \mod 4, \\ n+7, & r = n+5, \ n \equiv 0 \mod 4 \text{ and } n \equiv 1 \mod 4, \\ r+1, & r = n+6 \le r \le \Delta. \end{cases}$$

PROOF. By the definition of total graph,

$$\begin{aligned} V(T(G_n)) &= V(G_n) \cup E(G_n) \\ &= \{v\} \cup \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \\ &\quad \cup \{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le 2n\}. \end{aligned}$$

The vertices v and $\{e_i : 1 \le i \le n\}$ induces a clique of order K_{n+1} in $T(G_n)$. Thus, $\chi_r(T(G_n)) \ge n+1$. We divide the proof into some cases.

- $|N(e_i)| = \deg(e_i) = n+3$,
- $|N(s_i)| = \deg(s_i) = 5$,
- $|N(v)| = \deg(v) = 2n = \Delta$,
- $|N(u_i)| = \deg(u_i) = 4$,
- $|N(v_i)| = \deg(v_i) = 6$.

Case 1: For $1 \le r \le n$, the r-dynamic $(n+1)$-coloring is as follows:

For $1 \le i \le n$, assign the color c_i to e_i and c_{n+1} to v.

For $1 \le i \le n-1$, assign the color c_{i+1} to v_i and c_1 to v_n.

For $1 \le i \le n-3$, assign the color c_{i+3} to u_i and color the vertices u_{n-2}, u_{n-1} and u_n with the colors c_1, c_2 and c_3 respectively.

- Color the vertices $s_{2n}, s_2, s_4, s_6, \ldots, s_{2n-10}, s_{2n-8}, s_{2n-6}, s_{2n-4}, s_{2n-2}$ cyclically with the colors $c_5, c_6, c_7, \ldots, c_n, c_1, c_2, c_3, c_4$ (the order of the assigned colors is important).

- Color the vertices $s_1, s_3, s_5, \dots, s_{2n-13}, s_{2n-11}, s_{2n-9}, s_{2n-7}, s_{2n-5}, s_{2n-3}, s_{2n-1}$ cyclically with the colors $c_7, c_8, c_9, \dots, c_n, c_1, c_2, c_3, c_4, c_5, c_6$ (the order of the assigned colors is important).

Now, the r-adjacency condition is fulfilled hence $\chi_r(T(G_n)) \leq n+1$ since

$$n+1 \leq \chi_r(T(G_n)) \leq n+1 \Leftrightarrow \chi_r(T(G_n)) = n+1 \text{ for } 1 \leq r \leq n.$$

Case 2: For $r = n+1$, when n is even, the r-dynamic $(n+3)$-coloring is as follows:
For $1 \leq i \leq n$, assign the color c_i to e_i and c_{n+1} to v. $d(e_i) = n+3$, so in this case we need $n+1$ colors.

- Color the vertices v_i with the color c_{n+2} for $i = 1, 3, 5, \dots, n-1$ when n is even.
- Color the vertices v_i with the color c_{n+3} for $i = 2, 4, 6, \dots, n$ when n is even.

For $1 \leq i \leq n-3$, assign the color c_{i+3} to u_i and color the vertices u_{n-2}, u_{n-1}, u_n with the colors c_1, c_2, c_3 respectively.

- Color the vertices $s_{2n}, s_2, s_4, s_6, \dots, s_{2n-10}, s_{2n-8}, s_{2n-6}, s_{2n-4}, s_{2n-2}$ cyclically with the colors $c_5, c_6, c_7, \dots, c_n, c_1, c_2, c_3, c_4$ (the order of the assigned colors is important).
- Color the vertices $s_1, s_3, s_5, \dots, s_{2n-13}, s_{2n-11}, s_{2n-9}, s_{2n-7}, s_{2n-5}, s_{2n-3}, s_{2n-1}$ cyclically with the colors $c_7, c_8, c_9, \dots, c_n, c_1, c_2, c_3, c_4, c_5, c_6$ (the order of the assigned colors is important).

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+3$ for $r = n+1$ when n is even.

Case 3: For $r = n+1$, when n is odd. The r-dynamic $(n+4)$-coloring is as follows:
Assign the colors to the vertices e_i, v, u_i and s_i followed by case 2.

- Color the vertices v_i with the color c_{n+2} for $i = 1, 3, 5, \dots, n-2$, when n is odd.
- Color the vertices v_i with the color c_{n+3} for $i = 2, 4, 6, \dots, n-1$, when n is odd.

Now the vertex v_n is uncolored.

- Color the vertex v_n with the color c_{n+4}, when n is odd.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+4$ for $r = n+1$ when n is odd.

Case 4: For $r = n+2$, the r-dynamic $(n+5)$-coloring is as follows:
Assign the colors to the vertices e_i, v and u_i followed by case 2. $d(e_i) = n+3$, so in this case we need $n+2$ colors. We consider the subcases:

(a) When n is even,

- Color the vertices v_i with the color c_{n+2} for $i = 1, 3, 5, \ldots, n-1$, when n is even.
- Color the vertices v_i with the color c_{n+3} for $i = 2, 4, 6, \ldots, n$, when n is even.

Now neighbours of e_i getting $n+1$ colors. So we have to assign one new color to s_i.

- Color the vertices $s_{2n}, s_4, s_8, s_{12} \ldots, s_{2n-4}$ with the color c_{n+4}.
- Color the vertices $s_2, s_6, s_{10}, \ldots, s_{2n-6}, s_{2n-2}$ with the color c_{n+5}.

Now the vertices $s_1, s_3, s_5, s_7, \ldots, s_{2n-3}, s_{2n-1}$ are uncolored.

We have to assign the colors to the vertices $s_1, s_3, s_5, s_7, \ldots, s_{2n-3}, s_{2n-1}$ with the colors, if already exists.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+5$, $r = n+2$ when n is even.

(b) When n is odd,

- Color the vertices v_i with the color c_{n+2} for $i = 1, 3, 5, \ldots, n-2$, when n is odd.
- Color the vertices v_i with the color c_{n+3} for $i = 2, 4, 6, \ldots, n-1$, when n is odd.

Now the vertex v_n is uncolored.

- Color the vertex v_n with the color c_{n+4}.

Now neighbours of e_i getting $n+1$ colors. So we have to assign one new color to s_i.

- Color the vertex s_{2n} with the color c_{n+3}.
- Color the vertices $s_2, s_6, s_{10}, \ldots, s_{2n-8}, s_{2n-4}$ with the color c_{n+4}.
- Color the vertices $s_4, s_8, s_{12}, \ldots, s_{2n-6}, s_{2n-2}$ with the color c_{n+5}.

Now the vertices $s_1, s_3, s_5, s_7, \ldots, s_{2n-3}, s_{2n-1}$ are uncolored.

We have to assign the colors to the vertices $s_1, s_3, s_5, s_7, \ldots, s_{2n-3}, s_{2n-1}$ with the colors, if already exists.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+5$ for $r = n+2$ when n is odd.

Case 5: For $r = n+3$ and $n \equiv 0 \mod 6$, the r-dynamic $(n+6)$-coloring is as follows:

Assign the colors to the vertices e_i, v and u_i followed by case 2. $d(v) = 2n$, $d(e_i) = n+3$.

- Color the vertices v_i with the color c_{n+2}, for $i = 1, 4, 7, 10, \ldots, n-2$.
- Color the vertices v_i with the color c_{n+3}, for $i = 2, 5, 8, 11, \ldots, n-1$.
- Color the vertices v_i with the color c_{n+4}, for $i = 3, 6, 9, 12, \ldots, n$.

Now neighbours of e_i getting $n+1$ colors. So we have to assign two new colors to s_i.

- Color the vertices $s_{2n}, s_6, s_{12}, \ldots, s_{2n-6}$ with the color c_{n+3}.
- Color the vertices $s_1, s_5, s_9, s_{13}, \ldots, s_{2n-3}$ with the color c_{n+5}.
- Color the vertices $s_2, s_8, s_{14}, \ldots, s_{2n-4}$ with the color c_{n+4}.
- Color the vertices $s_3, s_7, s_{11}, \ldots, s_{2n-1}$ with the color c_{n+6}.
- Color the vertices $s_4, s_{10}, s_{16}, \ldots, s_{2n-2}$ with the color c_{n+2}.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+6$ for $r = n+3$ and $n \equiv 0 \mod 6$.

Case 6: For $r = n+3$ and $n \not\equiv 0 \mod 6$, the r-dynamic $(n+7)$-coloring is as follows:
Assign the colors to the vertices e_i, v and u_i followed by case 2. We subdivide the following cases:

(a) $n \equiv 1 \mod 6$.

- Color the vertices $v_1, v_3, v_5, \ldots, v_{n-2}$ with the color c_{n+2}.
- Color the vertices $v_2, v_4, v_6, \ldots, v_{n-1}$ with the color c_{n+3}.
- Color the vertex v_n with the color c_{n+4}.

$d(v) = 2n, d(e_i) = n+3$.

- Color the vertices s_{2n} with the color c_{n+3}.
- Color the vertices $s_2, s_6, s_{10}, \ldots, s_{2n-8}, s_{2n-4}$ with the color c_{n+4}
- Color the vertices $s_1, s_4, s_8, s_{12}, \ldots, s_{2n-6}, s_{2n-2}$ with the color c_{n+5}.
- Color the vertices $s_3, s_7, s_{11}, s_{15}, \ldots, s_{2n-7}, s_{2n-3}$ with the color c_{n+6}.
- Color the vertices $s_5, s_9, s_{13}, s_{17}, \ldots, s_{2n-5}, s_{2n-1}$ with the color c_{n+7}.

Now, an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+7$ for $r = n+3$ and $n \equiv 1 \mod 6$.

(b) $n \equiv 2 \mod 6$.

- Color the vertices $v_1, v_4, v_7, v_{10}, \ldots, v_{n-1}$ with the color c_{n+2}.
- Color the vertices $v_2, v_5, v_8, \ldots, v_{n-3}, v_n$ with the color c_{n+3}.
- Color the vertices $v_3, v_6, v_9, \ldots, v_{n-5}, v_{n-2}$ with the color c_{n+4}.
- Color the vertices $s_{2n}, s_8, s_{14}, s_{20}, \ldots, s_{2n-8}$ with the color c_{n+4}.

- Color the vertices $s_1, s_5, s_9, s_{13}, \ldots, s_{2n-7}, s_{2n-3}$ with the color c_{n+6}.
- Color the vertices s_2 and s_{2n-2} with the color c_{n+5}.
- Color the vertices $s_3, s_7, s_{11}, \ldots, s_{2n-5}, s_{2n-1}$ with the color c_{n+7}.
- Color the vertices $s_4, s_{10}, s_{16}, \ldots, s_{2n-12}, s_{2n-6}$ with the color c_{n+2}.
- Color the vertices $s_6, s_{12}, s_{18}, \ldots, s_{2n-10}, s_{2n-4}$ with the color c_{n+3}.

Now, an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n + 7$ for $r = n + 3$ and $n \equiv 2 \mod 6$.

(c) $n \equiv 3 \mod 6$.

- Color the vertices $v_1, v_3, v_5, \ldots, v_{n-2}$ with the color c_{n+2}.
- Color the vertices $v_2, v_4, v_6, \ldots, v_{n-1}$ with the color c_{n+3}.
- Color the vertex v_n with the color c_{n+4}.
- Color the vertex s_{2n} with the color c_{n+4}
- Color the vertices $s_4, s_8, s_{12}, \ldots, s_{2n-6}, s_{2n-2}$ with the color c_{n+5}
- Color the vertices $s_2, s_6, s_{10}, \ldots, s_{2n-8}, s_{2n-4}$ with the color c_{n+4}.
- Color the vertex s_1 with the color c_{n+5}
- Color the vertices $s_3, s_7, s_{11}, \ldots, s_{2n-7}, s_{2n-3}$ with the color c_{n+6}.
- Color the vertices $s_5, s_9, s_{13}, \ldots, s_{2n-5}, s_{2n-1}$ with the color c_{n+7}.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n + 7$ for $r = n + 3$ and $n \equiv 3 \mod 6$.

(d) $n \equiv 4 \mod 6$.

- Color the vertices $v_1, v_4, v_7, \ldots, v_{n-3}$ with the color c_{n+2}.
- Color the vertices $v_2, v_5, v_8, \ldots, v_{n-2}, v_n$ with the color c_{n+3}.
- Color the vertices $v_3, v_6, v_9, \ldots, v_{n-1}, v_n$ with the color c_{n+4}.
- Color the vertex s_{2n} with the color c_{n+4}
- Color the vertices $s_4, s_{10}, s_{16}, \ldots, s_{2n-10}, s_{2n-4}$ with the color c_{n+2}
- Color the vertices $s_1, s_5, s_9, \ldots, s_{2n-7}, s_{2n-3}$ with the color c_{n+6}.
- Color the vertices s_2 and s_{2n-2} with the color c_{n+5}
- Color the vertices $s_3, s_7, s_{11}, \ldots, s_{2n-5}, s_{2n-1}$ with the color c_{n+7}.

- Color the vertices $s_6, s_{12}, \ldots, s_{2n-14}, s_{2n-8}$ with the color c_{n+3}.
- Color the vertices $s_8, s_{14}, \ldots, s_{2n-6}, s_{2n}$ with the color c_{n+4}.

Now, an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r\left(T(G_n)\right) = n+7$ for $r = n+3$ and $n \equiv 4 \mod 6$.

(e) $n \equiv 5 \mod 6$.

- Color the vertices $v_1, v_3, v_5, \ldots, v_{n-2}$ with the color c_{n+2}.
- Color the vertices $v_2, v_4, v_6, \ldots, v_{n-1}$ with the color c_{n+3}.
- Color the vertex v_n with the color c_{n+4}.
- Color the vertex s_{2n} with the color c_{n+3}.
- Color the vertex s_1 with the color c_{n+5}.
- Color the vertices $s_2, s_6, s_{10}, \ldots, s_{2n-8}, s_{2n-4}$ with the color c_{n+4}.
- Color the vertices $s_3, s_7, s_{11}, \ldots, s_{2n-7}, s_{2n-3}$ with the color c_{n+6}.
- Color the vertices $s_4, s_8, s_{12}, \ldots, s_{2n-6}, s_{2n-2}$ with the color c_{n+5}.
- Color the vertices $s_5, s_9, s_{13}, \ldots, s_{2n-5}, s_{2n-1}$ with the color c_{n+7}.

Now, an easy check that the r-adjacency conditiont is fulfilled. Hence, $\chi_r\left(T(G_n)\right) = n+7$, for $r = n+3$ and $n \equiv 5 \mod 6$.

Case 7: For $r = n+4$ and $n \equiv 0 \mod 4$, the r-dynamic $(n+5)$-coloring is as follows: Assign the colors to the vertices e_i, v and u_i followed by case 2.

- Color the vertices $v_1, v_5, v_9, \ldots, v_{n-3}$ with the color c_{n+2}.
- Color the vertices $v_2, v_6, v_{10}, \ldots, v_{n-2}$ with the color c_{n+3}.
- Color the vertices $v_3, v_7, v_{11}, \ldots, v_{n-1}$ with the color c_{n+4}.
- Color the vertices $v_4, v_8, v_{12}, \ldots, v_n$ with the color c_{n+5}.
- Color the vertices $s_{2n}, s_5, s_8, s_{13}, s_{16}, \ldots, s_{2n-8}, s_{2n-3}$ with the color c_{n+3}.
- Color the vertices $s_1, s_4, s_9, s_{12}, s_{17}, s_{20}, \ldots, s_{2n-7}, s_{2n-4}$ with the color c_{n+5}.
- Color the vertices $s_2, s_7, s_{10}, s_{15}, s_{18}, \ldots, s_{2n-9}, s_{2n-6}, s_{2n-1}$ with the color c_{n+4}.
- Color the vertices $s_3, s_6, s_{11}, s_{14}, \ldots, s_{2n-10}, s_{2n-5}, s_{2n-2}$ with the color c_{n+2}.

Now, an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r\left(T(G_n)\right) = n+5$ for $r = n+4$ and $n \equiv 0 \mod 4$.

Case 8: For $r = n+4$ and $n \equiv 3 \mod 4$, the r-dynamic $(n+6)$-coloring is as follows:

- Color the vertices $v_1, v_5, v_9, \ldots, v_{n-2}$ with the color c_{n+2}.
- Color the vertices $v_2, v_6, v_{10}, \ldots, v_{n-1}$ with the color c_{n+3}.
- Color the vertices $v_3, v_7, v_{11}, \ldots, v_n$ with the color c_{n+4}.
- Color the vertices $v_4, v_8, v_{12}, \ldots, v_{n-3}$ with the color c_{n+5}.
- Color the vertices $s_{2n}, s_5, s_8, s_{13}, s_{16}, \ldots, s_{2n-6}$ with the color c_{n+3}.
- Color the vertices $s_1, s_4, s_9, s_{12}, \ldots, s_{2n-5}, s_{2n-2}$ with the color c_{n+5}.
- Color the vertices $s_2, s_7, s_{10}, \ldots, s_{2n-7}, s_{2n-4}$ with the color c_{n+4}.
- Color the vertices $s_3, s_6, s_{11}, s_{14}, \ldots, s_{2n-8}, s_{2n-3}$ with the color c_{n+2}.
- Color the vertex s_{2n-1} with the color c_{n+6}.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n + 6$ for $r = n + 4$ and $n \equiv 3 \mod 4$.

Case 9: For $r = n+4$ and $n \not\equiv 3 \mod 4$, the r-dynamic $(n+7)$-coloring is as follows:

Assign the colors to the vertices e_i, v and u_i followed by case 2. We subdivide the following cases:

(a) $n \equiv 1 \mod 4$.

- Color the vertices $v_1, v_5, v_9, \ldots, v_{n-4}$ with the color c_{n+2}.
- Color the vertices $v_2, v_6, v_{10}, \ldots, v_{n-3}, v_n$ with the color c_{n+3}.
- Color the vertices $v_3, v_7, v_{11}, \ldots, v_{n-2}$ with the color c_{n+4}.
- Color the vertices $v_4, v_8, v_{12}, \ldots, v_{n-1}$ with the color c_{n+5}.
- Color the vertices $s_{2n}, s_8, s_{16}, \ldots, s_{2n-10}$ with the color c_{n+4}.
- Color the vertices $s_1, s_5, s_9, \ldots, s_{2n-9}, s_{2n-5}$ with the color c_{n+6}.
- Color the vertices $s_2, s_{10}, s_{18}, \ldots, s_{2n-8}, s_{2n-1}$ with the color c_{n+5}.
- Color the vertices $s_3, s_7, s_{11}, \ldots, s_{2n-7}, s_{2n-3}$ with the color c_{n+7}.
- Color the vertices $s_4, s_{12}, s_{20}, \ldots, s_{2n-6}, s_{2n-2}$ with the color c_{n+2}.
- Color the vertices $s_6, s_{14}, \ldots, s_{2n-12}, s_{2n-4}$ with the color c_{n+3}.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n + 7$ for $r = n + 4$ and $n \equiv 1 \mod 4$.

(b) $n \equiv 2 \mod 4$.

- Color the vertices $v_1, v_5, v_9, \ldots, v_{n-5}, v_{n-1}$ with the color c_{n+2}.

- Color the vertices $v_2, v_6, v_{10}, \dots, v_{n-4}, v_n$ with the color c_{n+3}.
- Color the vertices $v_3, v_7, v_{11}, \dots, v_{n-3}$ with the color c_{n+4}.
- Color the vertices $v_4, v_8, v_{12}, \dots, v_{n-2}$ with the color c_{n+5}.
- Color the vertices $s_{2n}, s_8, s_{16}, \dots, s_{2n-4}$ with the color c_{n+4}.
- Color the vertices $s_1, s_5, s_9, \dots, s_{2n-7}, s_{2n-3}$ with the color c_{n+6}.
- Color the vertices $s_2, s_{10}, s_{18}, \dots, s_{2n-10}, s_{2n-2}$ with the color c_{n+5}.
- Color the vertices $s_3, s_7, s_{11}, \dots, s_{2n-5}, s_{2n-1}$ with the color c_{n+7}.
- Color the vertices $s_4, s_{12}, s_{20}, \dots, s_{2n-8}$ with the color c_{n+2}.
- Color the vertices $s_6, s_{14}, s_{22}, \dots, s_{2n-6}$ with the color c_{n+3}.

Now, an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+7$ for $r = n+4$ and $n \equiv 2 \mod 4$.

Case 10: For $r = n+5$, the r-dynamic $(n+6)$-coloring is as follows:

Assign the colors to the vertices e_i, v and u_i followed by case 2. We subdivide the following cases:
(a) $n \equiv 2 \mod 4$

- Color the vertices $v_1, v_6, v_{11}, \dots, v_{n-2}$ with the color c_{n+2}.
- Color the vertices $v_2, v_7, v_{12}, \dots, v_{n-1}$ with the color c_{n+3}.
- Color the vertices $v_3, v_8, v_{13}, \dots, v_n$ with the color c_{n+4}.
- Color the vertices $v_4, v_9, v_{14}, \dots, v_{n-4}$ with the color c_{n+5}.
- Color the vertices $v_5, v_{10}, v_{15}, \dots, v_{n-3}$ with the color c_{n+6}.
- Color the vertices $s_1, s_8, s_{11}, s_{18}, s_{21}, \dots, s_{2n-8}, s_{2n-5}$ with the color c_{n+4}.
- Color the vertices $s_2, s_5, s_{12}, s_{15}, \dots, s_{2n-4}, s_{2n-1}$ with the color c_{n+6}.
- Color the vertices $s_4, s_7, s_{14}, s_{17}, \dots, s_{2n-12}, s_{2n-9}, s_{2n-2}$ with the color c_{n+2}.
- Color the vertices $s_6, s_9, s_{16}, s_{19}, \dots, s_{2n-10}, s_{2n-7}$ with the color c_{n+3}.

Now, an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+6$, for $r = n+5$ and $n \equiv 2 \mod 4$

(b) $n \equiv 3 \mod 4$

- Color the vertices $v_1, v_6, v_{11}, \dots, v_{n-3}$ with the color c_{n+2}.
- Color the vertices $v_2, v_7, v_{12}, \dots, v_{n-2}$ with the color c_{n+3}.

- Color the vertices $v_3, v_8, v_{13}, \ldots, v_{n-1}$ with the color c_{n+4}.
- Color the vertices $v_4, v_9, v_{14}, \ldots, v_n$ with the color c_{n+5}.
- Color the vertices $v_5, v_{10}, v_{15}, \ldots, v_{n-4}$ with the color c_{n+6}.
- Color the vertices $s_{2n}, s_7, s_{10}, s_{17}, s_{20}, \ldots, s_{2n-11}, s_{2n-8}$ with the color c_{n+3}.
- Color the vertices $s_1, s_4, s_{11}, s_{14}, \ldots, s_{2n-7}, s_{2n-4}$ with the color c_{n+5}.
- Color the vertices $s_2, s_9, s_{12}, \ldots, s_{2n-9}, s_{2n-6}, s_{2n-1}$ with the color c_{n+4}.
- Color the vertices $s_3, s_6, s_{13}, s_{16}, \ldots, s_{2n-5}, s_{2n-2}$ with the color c_{n+6}.
- Color the vertices $s_5, s_8, s_{15}, s_{18}, \ldots, s_{2n-13}, s_{2n-10}, s_{2n-3}$ with the color c_{n+2}.

Now, an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+6$, for $r = n+5$ and $n \equiv 3 \mod 4$

Case 11: For $r = n+5$, the r-dynamic $(n+7)$-coloring is as follows:

Assign the colors to the vertices e_i, v and u_i followed by case 2.
We subdivide the following cases:
(a) $n \equiv 0 \mod 4$

- Color the vertices $v_1, v_6, v_{11}, \ldots, v_{n-5}$ with the color c_{n+2}.
- Color the vertices $v_2, v_7, v_{12}, \ldots, v_{n-4}, v_n$ with the color c_{n+3}.
- Color the vertices $v_3, v_8, v_{13}, \ldots, v_{n-3}$ with the color c_{n+4}.
- Color the vertices $v_4, v_9, v_{14}, \ldots, v_{n-2}$ with the color c_{n+5}.
- Color the vertices $v_5, v_{10}, v_{15}, \ldots, v_{n-1}$ with the color c_{n+6}.
- Color the vertices $s_{2n}, s_7, s_{10}, s_{17}, s_{20}, \ldots, s_{2n-5}$ with the color c_{n+4}.
- Color the vertices $s_1, s_4, s_{11}, s_{14}, \ldots, s_{2n-11}, s_{2n-8}$ with the color c_{n+6}.
- Color the vertices $s_2, s_9, s_{12}, \ldots, s_{2n-10}, s_{2n-2}$ with the color c_{n+5}.
- Color the vertices $s_3, s_6, s_{13}, s_{16}, \ldots, s_{2n-6}, s_{2n-3}$ with the color c_{n+2}.
- Color the vertices $s_5, s_8, s_{15}, s_{18}, \ldots, s_{2n-7}, s_{2n-4}$ with the color c_{n+3}.
- color the vertex s_{2n-1} with the color c_{n+7}.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n+7$, for $r = n+5$ and $n \equiv 0 \mod 4$

(b) $n \equiv 1 \mod 4$

- Color the vertices $v_1, v_6, v_{11}, \ldots, v_{n-1}$ with the color c_{n+2}.

- Color the vertices $v_2, v_7, v_{12}, \dots, v_{n-5}, v_n$ with the color c_{n+3}.
- Color the vertices $v_3, v_8, v_{13}, \dots, v_{n-4}$ with the color c_{n+4}.
- Color the vertices $v_4, v_9, v_{14}, \dots, v_{n-3}$ with the color c_{n+5}.
- Color the vertices $v_5, v_{10}, v_{15}, \dots, v_{n-2}$ with the color c_{n+6}.
- Color the vertices $s_{2n}, s_7, s_{10}, s_{17}, s_{20}, \dots, s_{2n-7}, s_{2n-4}$ with the color c_{n+4}.
- Color the vertices $s_1, s_4, s_{11}, s_{14}, \dots, s_{2n-10}, s_{2n-3}$ with the color c_{n+6}.
- Color the vertices $s_2, s_9, s_{12}, \dots, s_{2n-5}, s_{2n-2}$ with the color c_{n+5}.
- Color the vertices $s_3, s_6, s_{13}, s_{16}, \dots, s_{2n-11}, s_{2n-8}$ with the color c_{n+2}.
- Color the vertices $s_5, s_8, s_{15}, s_{18}, \dots, s_{2n-9}, s_{2n-6}$ with the color c_{n+3}.
- Color the vertex s_{2n-1} with the color c_{n+7}.

Now, an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = n + 7$, for $r = n + 5$ and $n \equiv 1 \mod 4$

Case 12: For $n + 6 \leq r \leq \Delta$, the r-dynamic $(r + 1)$-coloring is as follows:

Assign the colors to the vertices e_i, v and u_i followed by case 2.

Assign the suitable colors to the vertices v_i $(\forall i = 1, 2, \dots, n)$ from the color class $\{c_{n+2}, c_{n+3}, \dots, c_{r+1}\}$ and assign the colors to the vertices s_i $(\forall i = 1, 2, \dots, 2n)$ from the color class $\{c_{n+2}, c_{n+3}, \dots, c_{r+1}\}$ and an easy check shows that the r-adjacency condition is fulfilled. Hence, $\chi_r(T(G_n)) = r + 1$, for $n + 6 \leq r \leq \Delta$.

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