Applied Mathematics E-Notes, 16(2016), 244-250 © ISSN 1607-2510
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Restricted Factorial And A Remark On The Reduced Residue Classes*

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Received 26 March 2016

Abstract

In this paper we study the restricted factorial function $\widetilde{n}!!$ defined as the product of positive integers k not exceeding n and coprime to n. As a corollary, we consider the asymptotic behaviour of the ratio $\frac{A_n}{G_n}$, where A_n and G_n denote respectively the arithmetic and geometric means of all members of the least positive reduced set of residues modulo n.

1 Introduction

Among several questions concerning generalizations of the factorial function in [2], analogues of Stirling's approximation for generalized factorials is proposed. In the present paper we define the restricted factorial function for each integer $n \geqslant 1$ by

$$\widetilde{n}!! = \prod_{\substack{1\leqslant k\leqslant n \\ (k,n)=1}} k,$$

where (k, n) denotes the greatest common divisor of the integers k and n. We study the asymptotic growth of $\widetilde{n}!!$, and analogue to the well-known asymptotic relation

$$\log n! = n \log\left(\frac{n}{\mathrm{e}}\right) + O(\log n),$$

we obtain

$$\log \widetilde{n}!! = \phi(n) \log\left(\frac{n}{\mathrm{e}}\right) + O(\log n).$$

More precisely we prove the following.

THEOREM 1. We have

$$\log \widetilde{n}!! = \phi(n) \log\left(\frac{n}{\mathrm{e}}\right) + E(n),$$

where for $n \geqslant 7$ the remainder term $E(n)$ satisfies

$$-\frac{1}{2}\log\log n \leqslant E(n) \leqslant \frac{1}{2}\log\left(\frac{n}{\log n}\right).$$

*Mathematics Subject Classifications: 11A25, 11A07, 11N56.

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To obtain the above explicit bounds, we need some explicit bounds concerning $\log n!$, as follows.

LEMMA 2. For any integer $n \geqslant 1$ we have

$$\log n! = n\log n - n + \frac{1}{2}\log n + \frac{1}{2}\log(2\pi) + R(n), \tag{1}$$

where

$$0 \leqslant R(n) \leqslant \frac{1}{6n}. \tag{2}$$

Meanwhile, as an immediate consequence of Theorem 1 we obtain the following result.

COROLLARY 3. As $n \to \infty$, we have

$$\left(\widetilde{n!!}\right)^{\frac{1}{\phi(n)}} = \frac{n}{\mathrm{e}} + O(\log n \log\log n).$$

If we denote the arithmetic and geometric means of the positive real numbers $a_1, a_2, \ldots, a_n$, by $A(a_1, \ldots, a_n)$ and $G(a_1, \ldots, a_n)$, respectively, then the above corollary gives the asymptotic expansion of $G_n := G(\varrho_1, \ldots, \varrho_{\phi(n)})$, where $\mathcal{R}_n = \{\varrho_1, \ldots, \varrho_{\phi(n)}\}$ is the least positive reduced set of residues modulo n. By considering

$$A_n := A(\varrho_1, \ldots, \varrho_{\phi(n)}) = \frac{1}{\phi(n)} \sum_{\substack{1\leqslant k\leqslant n\\(k,n)=1}} k = \frac{n}{2},$$

we obtain the following.

COROLLARY 4. As $n \to \infty$, we have

$$\frac{A_n}{G_n} = \frac{\mathrm{e}}{2} + O\Big(\frac{\log n \log\log n}{n}\Big).$$

The ratio $\frac{\mathrm{e}}{2}$ appears surprisingly in studying the ratio of the arithmetic to the geometric means of some number theoretic sequences. For the sequence consisting of positive integers, Stirling's approximation for $n!$ implies (see [5] for more details)

$$\frac{A(1, \ldots, n)}{G(1, \ldots, n)} = \frac{\mathrm{e}}{2} + O\Big(\frac{\log n}{n}\Big).$$

Regarding to the sequence of prime numbers, in [6] we proved that

$$\frac{A(p_1, \ldots, p_n)}{G(p_1, \ldots, p_n)} = \frac{\mathrm{e}}{2} + O\Big(\frac{1}{\log n}\Big),$$

where p_n denotes the nth prime number. Moreover, in [3] we proved validity of the similar and more precise expansion

$$\frac{A(\gamma_1,\ldots,\gamma_n)}{G(\gamma_1,\ldots,\gamma_n)}=\frac{\mathrm{e}}{2}\left(1-\frac{1}{2\log n}-\frac{\log\log n}{2\log^2 n}-\frac{1}{2\log^2 n}\right)+O\Big(\frac{(\log\log n)^2}{\log^3 n}\Big),$$

where $0<\gamma_1<\gamma_2<\gamma_3<\cdots$ denote the consecutive ordinates of the imaginary parts of non-real zeros of the Riemann zeta-function, which is defined by $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$ for $\Re(s)>1$, and extended by analytic continuation to the complex plane with a simple pole at $s=1$.

On the other hand, the appearance of the similar limit value $\frac{\mathrm{e}}{2}$ in the above results is not trivial and a global property. As an example, we consider the asymptotic behaviour of the ratio under study for the values of the Euler function. By using the asymptotic expansions for $A(\phi(1),\ldots,\phi(n))$ and $G(\phi(1),\ldots,\phi(n))$ (see [13] for the arithmetic mean, and [7] for the geometric mean), we get

$$\frac{A(\phi(1),\ldots,\phi(n))}{G(\phi(1),\ldots,\phi(n))}=\frac{3\mathrm{e}}{\pi^2}\prod_p\left(1-\frac{1}{p}\right)^{-\frac{1}{p}}+O\Big(\frac{\log n}{n}\Big),$$

where the product runs over all primes. This gives a limit value different from $\frac{\mathrm{e}}{2}$, for the case of Euler function. More generally, we observe that the limit value of the ratio under study could be any arbitrary real number $\beta\geqslant 1$, as the following result confirms.

PROPOSITION 5. For each real number $\beta\geqslant 1$ there exists a real positive sequence with general term a_n such that

$$\lim_{n\to\infty}\frac{A(a_1,\ldots,a_n)}{G(a_1,\ldots,a_n)}=\beta.$$

Regarding to the case $\beta=1$, we show the following.

PROPOSITION 6. Assume that $a_n>0$ with $a_n\to\ell$ and $\ell>0$. Then

$$\lim_{n\to\infty}\frac{A(a_1,\ldots,a_n)}{G(a_1,\ldots,a_n)}=1.$$

We observe that Proposition 6 is not true for $\ell=0$. For instance, if we let $a_n=\frac{1}{n}$, then by using Stirling's approximation for $n!$, we obtain

$$\frac{A(a_1,\ldots,a_n)}{G(a_1,\ldots,a_n)}=\frac{1}{\mathrm{e}}\log n+O(1).$$

Finally, we note that if $d(n)$ denote the number of positive divisors of n, then in [4] we proved that for each fixed integer $m\geqslant 1$ one has

$$\frac{A(d(1),\ldots,d(n))}{G(d(1),\ldots,d(n))}=B^{-1}(\log n)^{1-\log 2}\left(1+\sum_{k=1}^{m}\frac{r_k}{\log^k n}+O\Big(\frac{1}{\log^{m+1}n}\Big)\right),$$

where B and the coefficients r_k are computable constants. This provides a number theoretic example for when the ratio $\frac{A}{G}$ tends to infinity.

2 Sums Over Reduced Residue Systems

To approximate $\log G_n$ we need to compute restricted summations running over the elements of $\mathcal{R}_n$. We follow the same method as in [1] to obtain the following.

PROPOSITION 7. Assume that f is an arbitrary arithmetic function. Then, we have

$$\sum_{\substack{1\leqslant k\leqslant n\\(k,n)=1}} f(k)=\sum_{d|n}\mu(d)\sum_{1\leqslant q\leqslant \frac{n}{d}} f(dq). \tag{3}$$

PROOF. The result is valid for $n=1$. We assume that $n>1$, and we use the known identity $\sum_{d|m}\mu(d)=[\frac{1}{m}]$ to write

$$\sum_{\substack{1\leqslant k\leqslant n\\(k,n)=1}} f(k)=\sum_{k=1}^{n-1} f(k)\left[\frac{1}{(k,n)}\right]=\sum_{k=1}^{n-1} f(k)\sum_{d|(k,n)}\mu(d)=\sum_{k=1}^{n-1}\sum_{d|k,d|n}\mu(d)f(k).$$

By taking $k=dq$, we get

$$\begin{aligned}\sum_{k=1}^{n-1}\sum_{d|k,d|n}\mu(d)f(k)&=\sum_{1\leqslant dq<n}\sum_{d|n}\mu(d)f(dq)\\&=\sum_{1\leqslant q<\frac{n}{d}}\sum_{d|n}\mu(d)f(dq)=\sum_{d|n}\mu(d)\sum_{1\leqslant q<\frac{n}{d}} f(dq).\end{aligned}$$

Now, we note that if $q=\frac{n}{d}$, then $f(dq)=f(n)$, and since $n>1$, we imply that $\sum_{d|n}\mu(d)f(n)=f(n)[\frac{1}{n}]=0$. Thus, we obtain (3), and the proof is complete.

3 Proofs

PROOF OF LEMMA 2. We apply Euler–Maclaurin summation formula (see [12]) with $f(k)=\log k$ to write

$$\log n! = n\log n - n + \frac{1}{2}\log n + 1 - \frac{1}{12} + \frac{1}{12n} + T_n,$$

where

$$T_n=\int_1^\infty \frac{B_2(\{x\})}{2x^2}\,\mathrm{d}x-\int_n^\infty \frac{B_2(\{x\})}{2x^2}\,\mathrm{d}x,$$

and $B_2(\{x\})$ is the Bernoulli function of order 2. Also, $\{x\}$ denotes the fractional part of the real x. Thus, we obtain

$$\log n! = n\log n - n + \frac{1}{2}\log n + C + \frac{1}{12n} - I,$$

with

$$C=\frac{11}{12}+\int_1^\infty \frac{B_2(\{x\})}{2x^2}\,\mathrm{d}x,$$

and

$$I = \int_n^\infty \frac{B_2(\{x\})}{2x^2}\,\mathrm{d}x.$$

Since $I \ll \frac{1}{n}$ as $n \to \infty$, we get

$$C = \lim_{n\to\infty}\Big(\log n! - \Big(n\log n - n + \frac{1}{2}\log n\Big)\Big) = \log \lim_{n\to\infty} D_n,$$

where

$$D_n = \frac{n!}{\left(\frac{n}{\mathrm{e}}\right)^n n^{\frac{1}{2}}}.$$

We apply Wallis product formula for π (see [14] for an elementary proof), to get

$$D^2 = \lim_{n\to\infty}\left(\frac{D_n D_n}{D_{2n}}\right)^2 = \lim_{n\to\infty}\frac{\prod_{k=1}^n (2k)^2}{\prod_{k=1}^n (2k-1)^2\,(2n+1)}\,\frac{2(2n+1)}{n} = 2\pi.$$

Thus, we obtain $D = \sqrt{2\pi}$, and consequently $C = \log D = \log\sqrt{2\pi}$. Also, we have

$$|I| \leqslant \int_n^\infty \frac{|B_2(\{x\})|}{2x^2}\,\mathrm{d}x \leqslant \frac{1}{12}\int_n^\infty \frac{\mathrm{d}x}{x^2} = \frac{1}{12n}.$$

This completes the proof.

PROOF OF THEOREM 1. By using (3) we have

$$\log \widetilde{n!!} = \sum_{\substack{1\leqslant k\leqslant n\\ (k,n)=1}} \log k = \sum_{d\mid n}\mu(d)\sum_{1\leqslant q\leqslant \frac{n}{d}}\log(dq) = \sum_{d\mid n}\mu(d)\left(\frac{n}{d}\log d + \log\left(\left(\frac{n}{d}\right)!\right)\right).$$

We apply the known relation $\sum_{d\mid n}\mu(d)\log d = -\Lambda(n)$, where $\Lambda(n)$ is the Mangoldt function, to obtain

$$\log \widetilde{n!!} = \phi(n)\log\left(\frac{n}{\mathrm{e}}\right) + E(n),$$

with

$$E(n) = \frac{1}{2}\Lambda(n) + \sum_{d\mid n}\mu(d)R\Big(\frac{n}{d}\Big),$$

and $R(n)$ is defined in (1). We have $0 \leqslant \Lambda(n) \leqslant \log n$. Also, by using the triangle inequality, and considering the bounds (2), we obtain

$$\left|\sum_{d\mid n}\mu(d)R\Big(\frac{n}{d}\Big)\right| \leqslant \sum_{d\mid n}\left|R\Big(\frac{n}{d}\Big)\right| \leqslant \frac{1}{6}\sum_{d\mid n}\frac{d}{n} = \frac{\sigma(n)}{6n} < \frac{1}{2}\log\log n,$$

where for deducing the last bound we use the inequality $\sigma(n) < 2.59n\log\log n$, which is valid for $n \geqslant 7$ (see [8]). Hence, for each $n \geqslant 7$ we get

$$-\frac{1}{2}\log\log n \leqslant E(n) \leqslant \frac{1}{2}\log\Big(\frac{n}{\log n}\Big).$$

This completes the proof.

PROOF OF COROLLARY 3. Theorem 1 implies that

$$(\widetilde{n!!})^{\frac{1}{\phi(n)}} = \Big(\frac{n}{\mathrm{e}}\Big)\mathrm{e}^{\frac{E(n)}{\phi(n)}}.$$

For any $n \geqslant 1$ we have $\phi(n) \leqslant n$. Also, the inequality

$$\phi(n) > \frac{n}{\mathrm{e}^{\gamma}\log\log n + \frac{2.50637}{\log\log n}},$$

is valid for $n \geqslant 3$ (see [10]). Thus, we get

$$\frac{E(n)}{\phi(n)} \ll \frac{\log n}{\frac{n}{\log\log n}} = \frac{\log n\log\log n}{n},$$

from which we obtain

$$\mathrm{e}^{\frac{E(n)}{\phi(n)}} = 1 + O\Big(\frac{\log n\log\log n}{n}\Big).$$

This completes the proof.

PROOF OF PROPOSITION 5. For each real number $\eta \geqslant 0$, we set $a_n = n^{\eta}$. It is known [9] that

$$\lim_{n\to\infty}\frac{A(a_1,\ldots,a_n)}{G(a_1,\ldots,a_n)} = \frac{\mathrm{e}^{\eta}}{\eta+1} := \ell(\eta),$$

say. We note that $\frac{\mathrm{d}}{\mathrm{d}\eta}\ell(\eta) = \ell(\eta)\frac{\eta}{\eta+1}$. Hence $\ell(\eta)$ is strictly increasing for $\eta \geqslant 0$. Also $\ell(0) = 1$ and $\lim_{\eta\to\infty}\ell(\eta) = \infty$. Thus, for any real number $\beta \geqslant 1$ there exists a real number $\eta \geqslant 0$ such that $\ell(\eta) = \beta$, as desired. This completes the proof.

PROOF OF PROPOSITION 6. For the sequence a_n addressed in the statement of theorem, it is known that $A(a_1,\ldots,a_n) \to \ell$ (see [11], page 80). Also, since $\log a_n \to \log\ell$ as $n \to \infty$, we obtain

$$\log G(a_1,\ldots,a_n) = A(\log a_1,\ldots,\log a_n) \to \log\ell,$$

and consequently, $G(a_1,\ldots,a_n) \to \ell$. This concludes the proof.

Acknowledgment. The author was supported by a grant of the University of Zanjan, research project number 9347.

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