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On Successive Coefficient Estimate For Certain Subclass Of Analytic Functions*

Trailokya Panigrahi†, Gangadharan Murugusundaramoorthy‡

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Abstract

The problem of estimating coefficient differences for the subclass $C \subset S$ of convex functions appears not to have been considered. The method of [11] seems not to be applicable to C. The object of the present paper is to give sharp estimates for the difference of coefficients of the univalent functions defined in the unit disk $\mathbb{U}$.

1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0, \quad f'(0) = 1$. Let S denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well-known that for $f \in \mathcal{A}$ the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{U})$$

is necessary and sufficient for starlikeness (and univalence) in the unit disk $\mathbb{U}$. Also, necessary and sufficient for $f \in \mathcal{A}$ to be convex in the unit disk is that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0.$$

These families of functions, denoted respectively by S^* and C, were discovered by Robertson [15](also see [4]).

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†Department of Mathematics, School of Applied Sciences, KIIT university, Bhubaneswar-751024, Orissa, India

‡Corresponding Author, Department of Mathematics, School of Advanced Science, VIT University, Vellore-632014, Tamil Nadu, India

For $n \geq 2$, Hayman [8] showed the difference of successive coefficients is bounded by an absolute constant i.e.

$$||a_{n+1}| - |a_n|| \leq A.$$

Using different technique, Milin [14] showed that $A < 9$. Ilina [9] improved this to $A < 4.26$. Further, Grispan [7] restricted to $A < 3.61$. For starlike function S^*, Leung [11] proved that the best possible bound $A = 1$. On the other hand, it is known that for the class S, A cannot be reduced to 1.When $n = 2$, Golusin [5, 6], Jenkins [10] and Duren [4] showed that for $f \subset S - 1 \leq |a_3| - |a_2| \leq 1.029\cdots$ and that both upper and lower bounds in (1) are sharp.

Recently, Darus and Ibrahim [3] introduced a differential operator

$$\mathcal{D}^{k,\alpha}_{\lambda,\delta} : \mathcal{A} \longrightarrow \mathcal{A}$$

by

$$\mathcal{D}^{k,\alpha}_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [n^\alpha + (n-1)n^\alpha \lambda]^k C(\delta, n) a_n z^n \tag{2}$$

where

$$C(\delta, n) = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}.$$

and $k, \alpha \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \lambda, \delta \geq 0$.

It should be remarked that the operator $\mathcal{D}^{k,\alpha}_{\lambda,\delta}$ is a generalization of many other linear operators studied by earlier researchers. Namely:

- for $\alpha = 1$ $\lambda = 0$, $\delta = 0$ or $\alpha = \delta = 0, \lambda = 1$, the operator $\mathcal{D}^{k,1}_{0,0} \equiv \mathcal{D}^{k,0}_{1,0} \equiv \mathcal{D}^k$ is the popular Salagean operator [17];
- for $k = 0$, the operator $\mathcal{D}^{0,\alpha}_{\lambda,\delta} \equiv \mathcal{D}^\delta$ has been studied by Ruscheweyh (see [16]);
- for $\alpha = 0$, $\delta = 0$, the operator $\mathcal{D}^{k,0}_{\lambda,0} \equiv \mathcal{D}^k_\lambda$ has been studied by Al-Oboudi (see [1]),
- for $\alpha = 0$, the operator $\mathcal{D}^{k,0}_{\lambda,\delta} \equiv \mathcal{D}^k_{\lambda,\delta}$ has been studied by Darus and Ibrahim (see [2]).

Making use of the differential operator $\mathcal{D}^{k,\alpha}_{\lambda,\delta}$, we introduce a new subclass of analytic functions as follows:

DEFINITION 1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}^{k,t}_{\lambda,\delta}(\alpha)$ if it satisfies the inequality

$$\Re\left(\frac{(1-t)z(\mathcal{D}^{k,\alpha}_{\lambda,\delta} f(z))' + tz(\mathcal{D}^{k+1,\alpha}_{\lambda,\delta} f(z))'}{(1-t)\mathcal{D}^{k,\alpha}_{\lambda,\delta} f(z) + t\, \mathcal{D}^{k+1,\alpha}_{\lambda,\delta} f(z)}\right) > 0 \tag{3}$$

where $z \in \mathbb{U};\ 0 \leq t \leq 1,\ k,\ \alpha \in \mathbb{N}_0,\ \lambda,$ and $\delta \geq 0$.

Note that by taking $t = k = \delta = 0$ and $t = \alpha = 1, k = \lambda = \delta = 0$ for the class $\mathcal{M}_{\lambda,\delta}^{k,t}(\alpha)$, we have the classes S^* and C respectively.

The purpose of the present study is to estimate the coefficient difference for the function class $\mathcal{M}_{\lambda,\delta}^{k,t}(\alpha)$ when $n = 2$ and $n = 3$.

2 Main Results

In order to derive our main results, we recall the following lemmas.

We denote by $\mathcal{P}$ a class of analytic functions in $\mathbb{U}$ with $p(0) = 1$ and $\Re(p(z)) > 0$.

LEMMA 1 (see [4]). Let the function $p \in \mathcal{P}$ be given by the series

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \quad (z \in \mathbb{U}). \tag{4}$$

Then, the sharp estimate

$$|c_k| \leq 2 \quad (k \in \mathbb{N}) \tag{5}$$

holds.

LEMMA 2 (cf. [12], also see [13]). Let the function $p \in \mathcal{P}$ be given by the series (4). Then

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{6}$$

for some x, $|x| \leq 1$ and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{7}$$

for some z and $|z| \leq 1$.

We now state and prove the following results.

THEOREM 1. Let f given by (1) be in the class $\mathcal{M}_{\lambda,\delta}^{k,t}(\alpha)$. If $\frac{3A_3}{4} \leq A_2 \leq \frac{3A_1}{2}$, then

$$||a_3| - |a_2|| \leq \frac{4A_1^2 + A_2^2}{4A_1^2 A_2}, \tag{8}$$

and

$$||a_4| - |a_3|| \leq \frac{A_2^2 + A_3^2}{A_2 A_3^2}, \tag{9}$$

where

$$A_1 = 2^{\alpha k}(1+\lambda)^k(\delta+1)\left[1 + (2^\alpha(1+\lambda) - 1)t\right],$$

$$A_2 = 3^{\alpha k}(1+2\lambda)^k \frac{(\delta+1)(\delta+2)}{2}\left[1 + (3^\alpha(1+2\lambda) - 1)t\right],$$

and

$$A_3 = 4^{\alpha k}(1+3\lambda)^k \frac{(\delta+1)(\delta+2)(\delta+3)}{6}\left[1 + (4^\alpha(1+3\lambda) - 1)t\right].$$

PROOF. Let the function $f(z)$ represented by (1) be in the class $\mathcal{M}_{\lambda,\delta}^{k,t}(\alpha)$. By geometric interpretation, there exists a function $h \in \mathcal{P}$ given by (4) such that

$$\frac{(1-t)z(\mathcal{D}_{\lambda,\delta}^{k,\alpha}f(z))' + tz(\mathcal{D}_{\lambda,\delta}^{k+1,\alpha}f(z))'}{(1-t)\mathcal{D}_{\lambda,\delta}^{k,\alpha}f(z) + t\,\mathcal{D}_{\lambda,\delta}^{k+1,\alpha}f(z)} = h(z). \tag{10}$$

Replacing $\mathcal{D}_{\lambda,\delta}^{k,\alpha}f(z)$, $\mathcal{D}_{\lambda,\delta}^{k+1,\alpha}f(z)$, $(\mathcal{D}_{\lambda,\delta}^{k,\alpha}f(z))'$, and $(\mathcal{D}_{\lambda,\delta}^{k+1,\alpha}f(z))'$ by their equivalent expressions and the equivalent expression for $h(z)$ in series (10), we have

$$\begin{aligned}
&(1-t)z\left\{1+\sum_{n=2}^{\infty} n[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)a_n z^{n-1}\right\} \\
&\qquad + tz\left\{1+\sum_{n=2}^{\infty} n[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k+1}C(\delta,n)a_n z^{n-1}\right\} \\
= \quad &(1-t)\left\{z+\sum_{n=2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k}C(\delta,n)a_n z^{n}\right\} \\
&\qquad + t\left\{z+\sum_{2}^{\infty}[n^{\alpha}+(n-1)n^{\alpha}\lambda]^{k+1}C(\delta,n)a_n z^{n}\right\}
\end{aligned} \tag{11}$$

Equating the coefficients of like power of z^2, z^3 and z^4 respectively on both sides of (11), we have

$$2A_1a_2 = c_1 + A_1a_2,$$

$$3A_2a_3 = c_2 + c_1A_1a_2 + A_2a_3,$$

$$4A_3a_4 = c_3 + A_1a_2c_2 + A_2a_3c_1 + A_3a_4,$$

where A_1, A_2 and A_3 are given in the statement of theorem.

After simplifying, we get

$$a_2 = \frac{c_1}{A_1}, \quad a_3 = \frac{c_2}{2A_2} + \frac{c_1^2}{2A_2}, \quad \text{and} \quad a_4 = \frac{c_3}{3A_3} + \frac{c_1c_2}{2A_3} + \frac{c_1^3}{6A_3}. \tag{12}$$

Since

$$||a_{n+1}| - |a_n|| \le |a_{n+1} - a_n|,$$

we need to consider $|a_3 - a_2|$ and $|a_3 - a_4|$.

Taking into account (12) and Lemma 2 we obtain

$$\begin{aligned}
|a_3 - a_2| &= \left|\frac{c_2}{2A_2} + \frac{c_1^2}{2A_2} - \frac{c_1}{A_1}\right| \\
&= \left|\frac{1}{2A_2}\left(\frac{c_1^2}{2} + \frac{x}{2}(4-c_1^2)\right) + \frac{c_1^2}{2A_2} - \frac{c_1}{A_1}\right| \\
&= \left|\frac{3}{4A_2}c_1^2 - \frac{c_1}{A_1} + \frac{x}{4A_2}(4-c_1^2)\right|.
\end{aligned} \tag{13}$$

We can assume without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c \quad (c \in [0,2])$ (see equation (5)). Applying triangle inequality and replacing $|x|$ by μ in the right hand side of (13) and using the inequality $A_2 \leq \frac{3A_1}{2}$, it reduces to

$$|a_3 - a_2| \leq \frac{c}{A_1} - \frac{3c^2}{4A_2} + \frac{4-c^2}{4A_2}\mu = F(c,\mu) \quad (0 \leq \mu = |x| \leq 1), \tag{14}$$

where

$$F(c,\mu) = \frac{c}{A_1} - \frac{3c^2}{4A_2} + \frac{4-c^2}{4A_2}\mu. \tag{15}$$

We assume that the upper bound for (14) occurs at an interior point of the $\{(\mu, c) : \mu \in [0.1] \text{ and } c \in [0,2]\}$. Differentiating (15) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = \frac{4-c^2}{4A_2}. \tag{16}$$

From (16) we observe that $\frac{\partial F}{\partial \mu} > 0$ for $0 < \mu < 1$ and for fixed c with $0 < c < 2$. Therefore $F(c,\mu)$ is an increasing function of μ, which contradicts our assumption that the maximum value of F occurs at an interior point of the set $\{(\mu, c) : \mu \in [0,1] \text{ and } c \in [0,2]\}$. So, fixed $c \in [0,2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c,\mu) = F(c,1) = G(c)(say).$$

Therefore replacing μ by 1 in (15), we obtain

$$G(c) = \frac{c}{A_1} + \frac{1}{A_2}(1-c^2), \tag{17}$$

$$G'(c) = \frac{1}{A_1} - \frac{2c}{A_2} \tag{18}$$

and

$$G''(c) = -\frac{2}{A_2} < 0.$$

For optimum value of $G(c)$, consider $G'(c) = 0$. It implies that $c = \frac{A_2}{2A_1}$. Therefore, the maximum value of $G(c)$ is $\frac{4A_1^2 + A_2^2}{4A_1^2 A_2}$ which occurs at $c = \frac{A_2}{2A_1}$. From the expression (17), we get

$$G_{\max} = G\left(\frac{A_2}{2A_1}\right) = \frac{4A_1^2 + A_2^2}{4A_1^2 A_2}. \tag{19}$$

Form (14) and (19), we have

$$|a_3 - a_2| \leq \frac{4A_1^2 + A_2^2}{4A_1^2 A_2},$$

which proves the assertion (8) of Theorem 2. Using the same technique, we will prove (9). From (12) and an application of Lemma 2 we have

$$\begin{aligned}|a_4 - a_3| &= \left|\frac{c_3}{3A_3} + \frac{c_1c_2}{2A_3} + \frac{c_1^3}{6A_3} - \frac{c_2}{2A_2} - \frac{c_1^2}{2A_2}\right| \\ &= \left|\frac{1}{12A_3}\{c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\}\right. \\ &\left. + \frac{c_1}{4A_3}\{c_1^2 + x(4-c_1^2)\} + \frac{c_1^3}{6A_3} - \frac{1}{4A_2}\{c_1^2 + x(4-c_1^2)\} - \frac{c_1^2}{2A_2}\right| \\ &= \left|\left|\frac{c_1^3}{2A_3} - \frac{3}{4A_2}c_1^2 + \frac{5c_1}{12A_3}(4-c_1^2)x - \frac{c_1(4-c_1^2)x^2}{12A_3}\right.\right. \\ &\left.\left. + \frac{1}{6A_3}(4-c_1^2)(1-|x|^2)z - \frac{1}{4A_2}(4-c_1^2)x\right|\right| \end{aligned} \tag{20}$$

As earlier, we assume without loss of generality that $c_1 = c$ with $0 \leq c \leq 2$. Applying triangle inequality and replacing $|x|$ by μ in the right hand side of (20) and using the fact that $A_3 \leq \frac{4A_2}{3}$, it reduces to

$$\begin{aligned}|a_4 - a_3| \leq& \frac{c^3}{2A_3} - \frac{3c^2}{4A_2} + \frac{5c}{12A_3}(4-c^2)\mu + \frac{c(4-c^2)\mu^2}{12A_3} \\ &+ \frac{1}{6A_3}(4-c^2)(1-\mu^2) + \frac{1}{4A_2}(4-c^2)\mu \\ =& H(c,\mu),\end{aligned} \tag{21}$$

where

$$\begin{aligned}H(c,\mu) =& \frac{c^3}{2A_3} - \frac{3c^2}{4A_2} + \frac{5c}{12A_3}(4-c^2)\mu + \frac{c(4-c^2)\mu^2}{12A_3} \\ &+ \frac{1}{6A_3}(4-c^2)(1-\mu^2) + \frac{1}{4A_2}(4-c^2)\mu.\end{aligned} \tag{22}$$

Suppose that $H(c,\mu)$ in (22) attains its maximum at an interior point (c,μ) of $[0,2]X[0,1]$. Differentiating (22) partially with respect to μ, we have

$$\begin{aligned}\frac{\partial H}{\partial \mu} &= \frac{5c}{12A_3}(4-c^2) + \frac{c(4-c^2)\mu}{6A_3} - \frac{1}{3A_3}(4-c^2)\mu + \frac{1}{4A_2}(4-c^2) \\ &= -\frac{1}{12A_3}(c^2-4)\left[c(5+2\mu) - 4\mu + \frac{3A_3}{A_2}\right].\end{aligned}$$

Now $\frac{\partial H}{\partial \mu} = 0$ which implies

$$c = \frac{4\left(\mu - \frac{3A_3}{4A_2}\right)}{2\mu + 5} < 0 \quad (0 < \mu < 1),$$

which is false since $c > 0$. Thus $H(c,\mu)$ attains its maximum on the boundary of $[0,2]X[0,1]$. Thus for fixed c, we have

$$\max_{0 \leq \mu \leq 1} H(c,\mu) = H(c,1) = J(c)(say).$$

Therefore, replacing μ by 1 in (22) and simplifying we get

$$J(c) = \frac{2c}{A_3} + \frac{1}{A_2} - \frac{c^2}{A_2}, \; J'(c) = \frac{2}{A_3} - \frac{2c}{A_2} \text{ and } J''(c) = -\frac{2}{A_2} < 0. \tag{23}$$

For an optimum value of $J(c)$, consider $J'(c) = 0$ which implies $c = \frac{A_2}{2A_1}$. Therefore, the maximum value of $J(c)$ occurs at $c = \frac{A_2}{A_3}$. From the expression (23) we obtain

$$J_{\max} = J\left(\frac{A_2}{A_3}\right) = \frac{A_2^2 + A_3^2}{A_2 A_3^2}. \tag{24}$$

From (21) and (24), we have

$$|a_4 - a_3| \leq \frac{A_2^2 + A_3^2}{A_2 A_3^2}.$$

The proof of Theorem 2 is thus completed.

Taking $t = \alpha = 1,\ \lambda = \delta = k = 0$ in theorem 2 we get Corollary 1.

COROLLARY 1. Let f given by (1) be in the class C. Then

$$||a_3| - |a_2|| \leq \frac{25}{38} \quad \text{and} \quad ||a_4| - |a_3|| \leq \frac{25}{38}.$$

Both the inequalities are sharp.

Putting $t = k = \delta = 0$ in theorem 2 we get Corollary 2.

COROLLARY 2. Let f given by (1) be in the class S^*. Then

$$||a_3| - |a_2|| \leq \frac{5}{4} \quad \text{and} \quad ||a_4| - |a_3|| \leq 2.$$

Both the inequalities are sharp.

Concluding Remark. Since (6) and (7) provide expressions only for the coefficients a_2, a_3 and a_4, the method in this paper cannot be used for $n > 4$. However it is possible that the bounds (8) and (9) holds for all $n > 2$.

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