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On The Edge Irregularity Strength Of Corona Product Of Graphs With Paths*

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Abstract

For a simple graph G, a vertex labeling $\phi : V(G) \to \{1, 2, \cdots, k\}$ is called k-labeling. The weight of an edge xy in G, denoted by $w_\pi(xy)$, is the sum of the labels of end vertices x and y, i.e. $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f, there is $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by $es(G)$. In this paper, we determine the exact value of edge irregularity strength of corona product of graphs with paths.

1 Introduction

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. By a *labeling* we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set or the edge-set, the labelings are called respectively *vertex labelings* or *edge labelings*. If the domain is $V(G) \cup E(G)$, then we call the labeling *total labeling*. Thus, for an edge k-labeling $\delta : E(G) \to \{1, 2, \cdots, k\}$ the associated weight of a vertex $x \in V(G)$ is

$$w_\delta(x) = \sum \delta(xy),$$

where the sum is over all vertices y adjacent to x.

Chartrand et al. [14] introduced edge k-labeling δ of a graph G such that $w_\delta(x) = \sum \delta(xy)$ for all vertices $x, y \in V(G)$ with $x \neq y$. Such labelings were called *irregular assignments* and the *irregularity strength* $s(G)$ of a graph G is known as the minimum k for which G has an irregular assignment using labels at most k. This parameter has attracted much attention [5, 8, 13, 15, 16, 20, 21].

Motivated by these papers, Baca et al. [11] defined a *vertex irregular total k-labeling* of a graph G to be a total labeling of $G, \psi : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$, such that

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the *total vertex-weights*

$$wt(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy)$$

are different for all vertices, that is, $wt(x) \neq wt(y)$ for all different vertices $x, y \in V(G)$. The *total vertex irregularity strength* of G, $tvs(G)$, is the minimum k for which G has a vertex irregular total k-labeling. They also defined the total labeling $\psi : V(G) \cup E(G) \to \{1, 2, \cdots, k\}$ to be an edge irregular total k-labeling of the graph G if for every two different edges xy and $x'y'$ of G one has

$$wt(xy) = \psi(x) + \psi(xy) + \psi(y) \neq wt(x'y') = \psi(x') + \psi(x'y') + \psi(y').$$

The *total edge irregularity strength*, $tes(G)$, is defined as the minimum k for which G has an edge irregular total k-labeling. Some results on the total vertex irregularity strength and the total edge irregularity strength can be found in [1, 2, 6, 9, 12, 18, 19, 21, 23, 24, 25].

The most complete recent survey of graph labelings is [17].

A vertex k-labeling $\phi : V(G) \to \{1, 2, \cdots, k\}$ is called an *edge irregular k-labeling* of the graph G if for every two different edges e and f, there is $w_\phi(e) \neq w_\phi(f)$, where the weight of an edge $e = xy \in E(G)$ is $w_\phi(xy) = \phi(x) + \phi(y)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, denoted by $es(G)$.

In [3], the authors estimated the bounds of the edge irregularity strength es and then determined its exact values for several families of graphs namely, paths, stars, double stars and Cartesian product of two paths. Mushayt [7] determined the edge irregularity strength of cartesian product of star, cycle with path P_2 and strong product of path P_n with P_2.

The following theorem established lower bound for the edge irregularity strength of a graph G.

THEOREM 1 ([3]). Let $G = (V, E)$ be a simple graph with maximum degree $\Delta = \Delta(G)$. Then

$$es(G) \geq \max\left\{ \left\lceil \frac{|E(G)| + 1}{2} \right\rceil, \Delta(G) \right\}.$$

In this paper, we determine the exact value of edge irregularity strength of corona graphs of path P_n with P_2, P_n with K_1 and P_n with S_m.

2 Main Results

The corona product of two graphs G and H, denoted by $G \odot H$, is a graph obtained by taking one copy of G (which has n vertices) and n copies $H_1, H_2, \ldots, H_n$ of H, and then joining the i-th vertex of G to every vertex in H_i.

The Corona product $P_n \odot P_m$ is a graph with the vertex set $V(P_n \odot P_m) = \{x_i, y_i^j : 1 \le i \le n, 1 \le j \le m\}$ and edge set

$$\begin{aligned} E(P_n \odot P_m) &= \{x_i x_{i+1} : 1 \le i \le n-1\} \cup \{x_i y_i^j : 1 \le i \le n, 1 \le j \le m\} \\ &\cup \{y_i^j y_i^{j+1} : 1 \le i \le n, 1 \le j \le m-1\}. \end{aligned}$$

In the next theorem, we determine the exact value of the edge irregularity strength of $P_n \odot P_2$.

THEOREM 2. For any integer $n \ge 2$. Then $es(P_n \odot P_2) = 2n + 1$.

PROOF. Let $P_n \odot P_2$ be a graph with the vertex set $V(P_n \odot P_2) = \{x_i, y_i^j : 1 \le i \le n, 1 \le j \le 2\}$ and the edge set

$$\begin{aligned} E(P_n \odot P_2) &= \{x_i x_{i+1} : 1 \le i \le n-1\} \\ &\cup \{x_i y_i^j : 1 \le i \le n, 1 \le j \le 2\} \\ &\cup \{y_i^1 y_i^2 : 1 \le i \le n\}. \end{aligned}$$

According to Theorem 1, we have that $es(P_n \odot P_2) \ge 2n$. Since every edge $E(P_n \odot P_2) \setminus \{x_i x_{i+1}\}$ for $1 \le i \le n-1$ are a part of complete graph K_3, therefore under every edge irregular labeling the smallest edge weight has to be at least 3 of said edges. Therefore the smallest edge weight 2 and the largest edge weight $4n$ will be of edges $x_i x_{i+1}$. For this there will be two pair of adjacent vertices such that one pair of adjacent vertices assign label 1, second pair of adjacent vertices assign label $2n$, then there will be two distinct edges having the same weight. Therefore $es(P_n \odot P_2) \ge 2n + 1$. To prove the equality, it suffices to prove the existence of an optimal edge irregular $(2n+1)$-labeling.

Let $\phi_1 : V(P_n \odot P_2) \to \{1, 2, \ldots, 2n+1\}$ be the vertex labeling such that

$$\phi_1(x_i) = 4\left\lceil \frac{i}{2} \right\rceil - 1 \text{ for } 1 \le i \le n$$

and

$$\phi_1(y_i^j) = 3\left\lceil \frac{i-1}{2} \right\rceil + \left\lceil \frac{i}{2} \right\rceil + j - 1 \text{ for } 1 \le i \le n \text{ and } 1 \le j \le 2.$$

Since

$$w_{\phi_1}(x_i x_{i+1}) = \phi_1(x_i) + \phi_1(x_{i+1}) = 4i + 2 \text{ for } 1 \le i \le n-1,$$

$$w_{\phi_1}(y_i^1 y_i^2) = \phi_1(y_i^1) + \phi_1(y_i^2) = 6\left\lceil \frac{i-1}{2} \right\rceil + 2\left\lceil \frac{i}{2} \right\rceil + 1 \text{ for} 1 \le i \le n \text{ and } 1 \le j \le 2,$$

and

$$w_{\phi_1}(x_i^1 y_i^j) = \phi_1(x_i) + \phi_1(y_i^j) = 3\left\lceil \frac{i-1}{2} \right\rceil + 5\left\lceil \frac{i}{2} \right\rceil + j - 2 \text{ for} 1 \le i \le n \text{ and } 1 \le j \le 2,$$

we see that the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ_1 is an optimal edge irregular $(2n+1)$-labeling. This completes the proof.

In Theorem 2, we determined the exact value of the edge irregularity strength of $P_n \odot P_m$ for $n \geq 2, m = 2$. We have try to find edge irregularity strength of $P_n \odot P_m$ for $n, m \leq 3$ but so far without success. So I conclude the following open problem.

OPEN PROBLEM. For the corona product $P_n \odot P_m$ for $n, m \leq 3$, determine the exact value of edge irregularity strength.

In the following theorem, we determine the exact value of the edge irregularity strength of $P_n \odot mK_1$.

THEOREM 3. For any integer $n \geq 2$ and $1 \leq j \leq m$. Then $es(P_n \odot mK_1) = \left\lceil \frac{n(m+1)}{2} \right\rceil$.

PROOF. Let $P_n \odot mK_1$ be a graph with the vertex set $V(P_n \odot mK_1) = \{x_i, y_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ and the edge set

$$E(P_n \odot mK_1) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\}.$$

By Theorem 1, it follows that $es(P_n \odot mK_1) \geq \left\lceil \frac{n(m+1)}{2} \right\rceil$. For the converse, we define a suitable edge irregular labeling

$$\phi_2 : V(P_n \odot mK_1) \to \left\{1, 2, \ldots, \left\lceil \frac{n(m+1)}{2} \right\rceil \right\}.$$

Case 1: Assume that n is even. We observe that

$$\phi_2(x_i) = \begin{cases} \frac{i-1}{2}(m+1)+1, & \text{if } i \text{ is odd,} \\ \frac{i}{2}(m+1), & \text{if } i \text{ is even,} \end{cases}$$

and

$$\phi_2(y_i^j) = \begin{cases} \frac{i-1}{2}(m+1)+j, & \text{if } i \text{ is odd and } 1 \leq j \leq m, \\ \frac{i-2}{2}(m+1)+j+1, & \text{if } i \text{ is even and } 1 \leq j \leq m. \end{cases}$$

Since

$$w_{\phi_2}(x_i x_{i+1}) = \phi_2(x_i) + \phi_2(x_{i+1}) = i(m+1)+1 \quad \text{for } 1 \leq i \leq n-1,$$

and

$$w_{\phi_2}(x_i y_i^j) = \phi_2(x_i) + \phi_2(y_i^j) = (i-1)(m+1)+j+1 \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq m,$$

we see that the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ_2 is an optimal edge irregular $\lceil \frac{n(m+1)}{2} \rceil$-labeling.

Case 2: Assume that n is odd. We observe that

$$\phi_2(x_i) = \begin{cases} \frac{i-1}{2}(m+1)+1, & \text{if } 1 \leq i \leq n-1 \text{ and } i \text{ odd,} \\ \frac{i}{2}(m+1), & \text{if } 1 \leq i \leq n-1 \text{ and } i \text{ even,} \\ \left\lceil \frac{n(m+1)}{2} \right\rceil, & \text{if } i = n, \end{cases}$$

$$\phi_2(y_i^j) = \begin{cases} \frac{i-1}{2}(m+1)+j, & \text{if } 1 \le i \le n-1,\ i \text{ is odd, and } 1 \le j \le m, \\ \frac{i-2}{2}(m+1)+j+1, & \text{if } 1 \le i \le n-1,\ i \text{ is even, and } 1 \le j \le m, \end{cases}$$

$$\phi_2(y_n^j) \in \left\{ \left\lceil \frac{n(m+1)}{2} \right\rceil, \left\lceil \frac{n(m+1)}{2} \right\rceil - 1, \cdots, \left\lceil \frac{n(m+1)}{2} \right\rceil - m \right\} \Big\backslash \left\{ \frac{n-1}{2}(m+1) \right\}.$$

Since

$$w_{\phi_2}(x_i x_{i+1}) = \phi_2(x_i) + \phi_2(x_{i+1}) = i(m+1)+1 \text{ for } 1 \le i \le n-2,$$

$$\begin{aligned} w_{\phi_2}(x_{n-1}x_n) &= \phi_2(x_{n-1}) + \phi_2(x_n) \\ &= \frac{n-1}{2}(m+1) + \left\lceil \frac{n(m+1)}{2} \right\rceil \text{ for } 1 \le i \le n-1 \text{ and } 1 \le j \le m, \end{aligned}$$

$$w_{\phi_2}(x_i y_i^j) = \phi_2(x_i) + \phi_2(y_i^j) = (i-1)(m+1)+j+1 \text{ for } 1 \le i \le n-1 \text{ and } 1 \le j \le m,$$

and

$$\begin{aligned} w_{\phi_2}(x_n y_n^j) &= \phi_2(x_n) + \phi_2(y_n^j) \\ &= \left\{ \left\lceil \frac{n(m+1)}{2} \right\rceil, \left\lceil \frac{n(m+1)}{2} \right\rceil - 1, \ldots, \left\lceil \frac{n(m+1)}{2} \right\rceil - m \right\} \\ &\quad \Big\backslash \left\{ \frac{n-1}{2}(m+1) \} + \left\lceil \frac{n(m+1)}{2} \right\rceil \right\}, \end{aligned}$$

we see that the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ_2 is an optimal edge irregular $\lceil \frac{n(m+1)}{2} \rceil$-labeling. This completes the proof.

Let P_n be a path of order n and S_m be a star of order $m+1$ with z as a central vertex. The Corona product $P_n \odot S_m$ is a graph with the vertex set

$$V(P_n \odot S_m) = \{x_i, y_i^j, z_i : 1 \le i \le n, 1 \le j \le m\}$$

and the edge set

$$E(P_n \odot S_m) = \{x_i x_{i+1} : 1 \le i \le n-1\} \cup \{x_i y_i^j, x_i z_i, z_i y_i^j : 1 \le i \le n, 1 \le j \le m\}.$$

Clearly, $|V(P_n \odot S_m| = n(m+2)$ and $|E(P_n \odot S_m)| = 2n(m+1) - 1$. The following theorem gives the exact value of the edge irregularity strength for $P_n \odot S_m$.

THEOREM 4. For $n \ge 2$ and $m \ge 3$. Then

$$es(P_n \odot S_m) = nm + n + 1.$$

PROOF. According to Theorem 1, we have that $es(P_n \odot S_m) \ge nm + n$. Since the edges $x_i y_i^j, x_i z_i$ and $z_i y_i^j$ are parts of complete graph K_3, therefore under every edge irregular $(nm+n)-$labeling, the smallest edge weight has to be at least 3. Therefore, the edges $x_i x_{i+1}$ attain the smallest and largest edge weights 2 and $2(nm+n)$, respectively. This is not possible under the every edge irregular $(nm+n)-$labeling. Therefore the largest vertex label will be $nm+n+1$. This implies that $es(P_n \odot S_m) \ge nm+n+1$.

To prove the equality, it suffices to prove the existence of an optimal edge irregular $(nm+n+1)-$labeling.

Let $\phi_3 : V(P_n \odot S_m) \to \{1, 2, \ldots, nm+n+1\}$ be the vertex labeling such that

$$\phi_3(x_i) = 2\left\lfloor \frac{i}{2} \right\rfloor (m+1) + 1 \quad \text{for } 1 \le i \le n,$$

$$\phi_3(y_i^j) = m(i-1) + i + j \quad \text{for } 1 \le i \le n, 1 \le j \le m,$$

and

$$\phi_3(z_i) = 2\left\lfloor \frac{i}{2} \right\rfloor (m+1) - m \quad \text{for } 1 \le i \le n.$$

Since

$$w_{\phi_3}(x_i x_{i+1}) = \phi_3(x_i) + \phi_3(x_{i+1}) = 2i(m+1) + 2 \quad \text{for } 1 \le i \le n-1,$$

and since

$$w_{\phi_3}(x_i z_i) = \phi_3(x_i) + \phi_3(z_i) = 2i(m+1) - m + 1,$$

$$w_{\phi_3}(x_i y_i^j) = \phi_3(x_i) + \phi_3(y_i^j) = (m+1)\left(2\left\lfloor \frac{i}{2} \right\rfloor + i\right) - m + j + 1$$

and

$$w_{\phi_3}(z_i y_i^j) = \phi_3(z_i) + \phi_3(y_i^j) = (m+1)\left(2\left\lceil \frac{i}{2} \right\rceil + i\right) - 2m + j,$$

for $1 \le i \le n$ and $1 \le j \le m$, we see that the edge weights are distinct for all pairs of distinct edges. Thus, the vertex labeling ϕ_3 is an optimal edge irregular $(nm+n+1)$-labeling. This completes the proof.

3 Conclusion

In this paper, we discussed the new graph characteristic, the edge irregularity strength, as a modification of the well-known irregularity strength, total edge irregularity strength and total vertex irregularity strength (see [3, 7]). We obtained the precise values for edge irregularity strength of corona graphs of path P_n with P_2, P_n with K_1 and P_n with S_m. It seems to be a very challenging problem to find the exact value for the edge irregularity strength of families of graphs.

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