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Laplacian Spectral Radius And Some Hamiltonian Properties Of Graphs*

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Abstract

Using an upper bound for the Laplacian spectral radius of graphs obtained by Shu et al., we in this note present sufficient conditions which are based on the Laplacian spectral radius for some Hamiltonian properties of graphs.

1 Introduction

We consider only finite undirected graphs without loops and multiple edges. Notation and terminology not defined here follow those in [2]. For a graph $G = (V, E)$, we use n and e to denote its order $|V|$ and size $|E|$, respectively. We use $\delta = d_1 \leq d_2 \leq ... \leq d_n = \Delta$ to denote the degree sequence of a graph. A cycle C in a graph G is called a Hamiltonian cycle of G if C contains all the vertices of G. A graph G is called Hamiltonian if G has a Hamiltonian cycle. A path P in a graph G is called a Hamiltonian path of G if P contains all the vertices of G. A graph G is called traceable if G has a Hamiltonian path. A graph G is called Hamilton-connected if for each pair of vertices in G there is a Hamiltonian path between them. The eigenvalues of a graph G are defined as the eigenvalues of its adjacency matrix $A(G)$. The largest eigenvalue of a graph G is called the spectral radius of G. The Laplacian eigenvalues of a graph G are defined as the eigenvalues of the matrix $L(G) := D(G) - A(G)$, where $D(G)$ is the diagonal matrix $diag(d_1, d_2, ..., d_n)$ and $A(G)$ is the adjacency matrix of G. The largest Laplacian eigenvalue of a graph G, denoted $\mu(G)$, is called the Laplacian spectral radius of G.

In this note, we, using an upper bound for the Laplacian spectral radius of graphs obtained by Shu et al. in [3], will present sufficient conditions which are based on the Laplacian spectral radius for some Hamiltonian properties of graphs. The results are as follows.

THEOREM 1. Let G be a connected graph with order $n \geq 3$, size e and minimum degree δ. If $(2\delta + 1)^2 + 4(f_1(n) - 2\delta(e + 1)) \geq 0$ and

$$\mu > \frac{(2\delta + 1) + \sqrt{(2\delta + 1)^2 + 4(f_1(n) - 2\delta(e + 1))}}{2},$$

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then G is Hamiltonian, where $f_1(n) = \left(5(n-1)^3 + 8(n-2)^3\right)/8$.

THEOREM 2. Let G be a connected graph with order $n \geq 2$, size e and minimum degree δ. If $(2\delta+1)^2 + 4(f_2(n) - 2\delta(e+1)) \geq 0$ and

$$\mu > \frac{(2\delta+1) + \sqrt{(2\delta+1)^2 + 4(f_2(n) - 2\delta(e+1))}}{2},$$

then G is traceable, where $f_2(n) = (n(n-2)^2 + 8(n-3)^3 + 4(n-2)(n-1)^2)/8$.

THEOREM 3. Let G be a connected graph with order $n \geq 3$, size e and minimum degree δ. If $(2\delta+1)^2 + 4(f_3(n) - 2\delta(e+1)) \geq 0$ and

$$\mu > \frac{(2\delta+1) + \sqrt{(2\delta+1)^2 + 4(f_3(n) - 2\delta(e+1))}}{2},$$

then G is Hamilton-connected, where $f_3(n) = ((n-2)n^2 + 8(n-3)^3 + 4n(n-1)^2)/8$.

2 Lemmas

In order to prove the theorems above, we need the following results as our lemmas.

LEMMA 1. Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If

$$d_k \leq k < \frac{n}{2} \Longrightarrow d_{n-k} \geq n-k,$$

then G is Hamiltonian.

LEMMA 2. Let G be a graph of order $n \geq 2$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If

$$d_k \leq k-1 \leq \frac{n}{2} - 1 \Longrightarrow d_{n+1-k} \geq n-k,$$

then G is traceable.

LEMMA 3. Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. If

$$2 \leq k \leq \frac{n}{2},\ d_{k-1} \leq k \Longrightarrow d_{n-k} \geq n-k+1,$$

then G is Hamilton-connected.

LEMMA 4 ([3]). Let G be a connected graph of order n with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Then

$$\mu(G) \leq d_1 + \frac{1}{2} + \sqrt{\left(d_1 - \frac{1}{2}\right)^2 + \sum_{i=1}^{n} d_i(d_i - d_1)},$$

the equality holds if and only if G is a regular bipartite graph.

Notice that Lemmas 1, 2, and 3 above are respectively Corollary 3 on Page 209, Corollary 6 on Page 210, and Theorem 12 on Page 218 in [1]. Next, we will present the proofs of Theorems 1–3.

3 Proofs

In this section, we prove Theorems 1–3.

PROOF of THEOREM 1. Let G be a graph satisfying the conditions in Theorem 1. Suppose that G is not Hamiltonian. Then, from Lemma 1, there exists an integer $k < \frac{n}{2}$ such that $d_k \leq k$ and $d_{n-k} \leq n-k-1$. Obviously, $k \geq 1$ and $d_k \geq 1$. Then, from Lemma 4, we have that

$$\mu \leq d_1 + \frac{1}{2} + \sqrt{\left(d_1 - \frac{1}{2}\right)^2 + \sum_{i=1}^{n} d_i(d_i - d_1)},$$

Thus

$$\mu^2 - \mu(2\delta + 1) + 2\delta(1 + e) \leq \sum_{i=1}^{n} d_i^2.$$

Notice that

$$\begin{aligned} \sum_{i=1}^{n} d_i^2 &\leq k^3 + (n-2k)(n-k-1)^2 + k(n-1)^2 \\ &\leq \left(\frac{n-1}{2}\right)^3 + (n-2)^3 + \frac{(n-1)^3}{2} = \frac{5(n-1)^3 + 8(n-2)^3}{8}. \end{aligned}$$

Set

$$f_1(n) := \frac{5(n-1)^3 + 8(n-2)^3}{8}.$$

Hence

$$\mu^2 - \mu(2\delta + 1) + 2\delta(1 + e) - f_1(n) \leq 0.$$

Since $(2\delta + 1)^2 + 4(f_1(n) - 2\delta(e + 1)) \geq 0$, we can solve the inequality and get

$$\mu \leq \frac{(2\delta + 1) + \sqrt{(2\delta + 1)^2 + 4(f_1(n) - 2\delta(e + 1))}}{2},$$

which is a contradiction. This completes the proof of Theorem 1.

PROOF of THEOREM 2. Let G be a graph satisfying the conditions in Theorem 2. Suppose that G is not traceable. Then, from Lemma 2, there exists an integer $k \leq \frac{n}{2}$ such that $d_k \leq k-1$ and $d_{n+1-k} \leq n-k-1$. Obviously, $k \geq 2$ and $d_k \geq 1$. Then, from Lemma 4, we have that

$$\mu \leq d_1 + \frac{1}{2} + \sqrt{\left(d_1 - \frac{1}{2}\right)^2 + \sum_{i=1}^{n} d_i(d_i - d_1)},$$

Thus

$$\mu^2 - \mu(2\delta + 1) + 2\delta(1 + e) \leq \sum_{i=1}^{n} d_i^2.$$

Notice that

$$\begin{aligned}\sum_{i=1}^{n} d_i^2 &\leq k(k-1)^2+(n-2k+1)(n-k-1)^2+(k-1)(n-1)^2 \\ &\leq \frac{n}{2}\left(\frac{n-2}{2}\right)^2+(n-3)^3+\frac{(n-2)(n-1)^2}{2} \\ &= \frac{n(n-2)^2+8(n-3)^3+4(n-2)(n-1)^2}{8}.\end{aligned}$$

Set

$$f_2(n) := \frac{n(n-2)^2+8(n-3)^3+4(n-2)(n-1)^2}{8}.$$

Hence

$$\mu^2-\mu(2\delta+1)+2\delta(1+e)-f_2(n)\leq 0.$$

Since $(2\delta+1)^2+4(f_2(n)-2\delta(e+1))\geq 0$, we can solve the inequality and get

$$\mu \leq \frac{(2\delta+1)+\sqrt{(2\delta+1)^2+4(f_2(n)-2\delta(e+1))}}{2},$$

which is a contradiction. This completes the proof of Theorem 2.

PROOF of THEOREM 3. Let G be a graph satisfying the conditions in Theorem 3. Suppose that G is not Hamilton-connected. Then, from Lemma 3, there exists an integer k such that $2\leq k\leq \frac{n}{2}$, $d_{k-1}\leq k$, and $d_{n-k}\leq n-k-1$. Obviously, $d_{k-1}\geq 1$. Then, from Lemma 4, we have that

$$\mu \leq d_1+\frac{1}{2}+\sqrt{\left(d_1-\frac{1}{2}\right)^2+\sum_{i=1}^{n} d_i(d_i-d_1)},$$

Thus

$$\mu^2-\mu(2\delta+1)+2\delta(1+e)\leq \sum_{i=1}^{n} d_i^2.$$

Notice that

$$\begin{aligned}\sum_{i=1}^{n} d_i^2 &\leq (k-1)k^2+(n-2k+1)(n-k-1)^2+k(n-1)^2 \\ &\leq \left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right)^2+(n-3)^3+\frac{n(n-1)^2}{2} \\ &= \frac{(n-2)n^2+8(n-3)^3+4n(n-1)^2}{8}.\end{aligned}$$

Set

$$f_3(n) := \frac{(n-2)n^2+8(n-3)^3+4n(n-1)^2}{8}.$$

Hence

$$\mu^2 - \mu(2\delta + 1) + 2\delta(1 + e) - f_3(n) \leq 0.$$

Since $(2\delta + 1)^2 + 4(f_3(n) - 2\delta(e + 1)) \geq 0$, we can solve the inequality and get

$$\mu \leq \frac{(2\delta + 1) + \sqrt{(2\delta + 1)^2 + 4(f_3(n) - 2\delta(e + 1))}}{2},$$

which is a contradiction. This completes the proof of Theorem 3.

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