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The Derivative Of A Finite Continued Fraction*

Jussi Malila†

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Abstract

The derivative of a finite continued fraction of a complex variable is derived by presenting the continued fraction as a component of a finite composition of $\hat{\mathbb{C}}^2 \to \hat{\mathbb{C}}^2$ linear fractional transformations of analytic functions. Connections to previous work and possible applications of the deduced formula are briefly discussed.

1 Introduction

Continued fractions

$$\mathop{\mathrm{K}}_{k=n}^{\infty}\left(\frac{a_k}{b_k}\right) = b_{n-1} + \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{b_{n+1} + \cfrac{a_{n+2}}{\ddots}}} \tag{1}$$

occur frequently in applications due to close connections between continued fractions and second-order difference and differential equations [1]. Although it has been known since the pioneering work of Euler that one can convert a continued fraction into a power series, implying analyticity in the whole domain of definition as long as the elements a_k and b_k are analytic, it is difficult to give general formulae for the derivatives of a continued fraction with respect to its argument. In some cases this can be accomplished by utilizing a connection between a given continued fraction and a special function (e.g. [2], see [3] for further possibilities). So far most studies have concentrated on the more mathematically interesting case of an infinite continued fraction. In applications, however, boundary conditions—or computational limitations—lead to the truncation of the continued fraction (1) after a finite number of levels, resulting in a finite continued fraction $\mathrm{K}_{k=n}^{N}(a_k/b_k)$. Here we derive a general formula for the derivative of this finite continued fraction by presenting it as a finite composition of linear fractional transformation of analytic functions. We then briefly discuss the connections between the deduced formula and partial derivatives with respect to the elements a_k and b_k.

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†Department of Applied Physics, University of Eastern Finland, P. O. Box 1627, FI-70211 Kuopio, Finland

2 Preliminaries and the Main Result

We consider a finite continued fraction

$$\mathop{\mathrm{K}}_{k=n}^{N}\left(\frac{a_k}{b_k}\right) = \cfrac{a_n}{b_n + \cfrac{a_{n+1}}{b_{n+1} + \cfrac{a_{n+2}}{\ddots \, b_{N-1} + \cfrac{a_N}{b_N}}}}, \tag{2}$$

where a_k and b_k are functions of a complex argument z that we suppress for brevity. This finite continued fraction is also known as the N*th approximant* of the infinite continued fraction (1). (We neglect the term b_{n-1} for simplicity, which does not affect the generality of our results.) For $N - n > \ell \geq 1$, the expression $\mathrm{K}_{k=n+\ell}^{N}(a_k/b_k)$ is known as the ℓ*th tail* of the finite continued fraction (2).

Now let $\{a_k\}_{k\geq n}$ and $\{b_k\}_{k\geq n}$ be two sequences of complex-valued analytic functions with domains Ψ and $\Omega \subset \mathbb{C}$, respectively. We define a third sequence of functions $\{g_k\}_{k\geq n}$ as

$$g_k(z,\zeta) = (g_{k,1}(z), g_{k,2}(z,\zeta)) = \left(z, \frac{a_k(z)}{b_k(z)+\zeta}\right), \tag{3}$$

with domain $G \subset \Psi \cap \Omega \times B(0,R_G) \subset \hat{\mathbb{C}}^2$ such that $g_k(G) \subseteq G$ for all $k = n, \ldots, N$; here $B(0,R_G)$ is an open disk with radius $R_G < \infty$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We set the (finite) $G \to G$ composite function

$$f_{n\to N}(z,\zeta) = (f_{n\to N,1}, f_{n\to N,2}) := g_n \circ g_{n+1} \circ \cdots \circ g_N(z,\zeta), \tag{4}$$

where subscript $n \to N$ indicates the extent of the composition.

THEOREM 1. Assuming that $a_\ell(z), b_\ell(z) + g_{\ell+1,2}(z,\zeta) \neq 0$ for $\ell = n, \ldots, N-1$ and $b_N(z) \neq 0$ for all $(z,\zeta) \in G$, it follows that

(i) $f_{n\to N}$ is analytic

(ii) $f_{n\to N,2}$ evaluated at $(z,0)$ equals the finite continued fraction (2).

PROOF. Claim (i) follows since $g_{k,1}$ and $g_{k,2}$ are $\mathbb{C} \to \mathbb{C}$ linear fractional transformations of analytic functions that are bounded and continuous for every $(z,\zeta) \in G$ and $k = n, \ldots, N$. Thus $f_{n\to N}$ is also analytic as a finite composition of analytic functions [4, Sec. 2.1].

Now $f_{n\to N,2}(z,0) = \mathrm{K}_{k=n}^{N}(a_k/b_k)$ is a well-defined $G \to \mathbb{C}$ function corresponding to the standard definition of the (nth tail of a) continued fraction [1, Sec. I.1], thus satisfying claim (ii).

LEMMA (Chain rule). The chain rule for $f_{n\to N,2}$ reads as

$$\frac{\partial f_{n\to N,2}}{\partial z} = \sum_{j=n}^{N}\left(\prod_{m=n+1}^{j} \frac{\partial g_{m-1,2}}{\partial g_{m,2}}\right)\frac{\partial g_{j,2}}{\partial z}. \tag{5}$$

PROOF. From Theorem 1 all $g_{k,2}$ are analytic and $\partial g_k/\partial(\bar{z},\bar{\zeta}) = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$. Equation (5) is most conveniently proven by induction; the argument is essentially the same as that of Gill [5], but is repeated here for the convenience of the reader. First, letting $N = n+1$, the result follows directly from the usual chain rule:

$$\begin{aligned}P_1 : \frac{\partial f_{n\to N,2}}{\partial z} &= \frac{\partial g_{n,2}}{\partial g_{n+1,1}}\frac{\partial g_{n+1,1}}{\partial z} + \frac{\partial g_{n,2}}{\partial g_{n+1,2}}\frac{\partial g_{n+1,2}}{\partial z} \\ &= \frac{\partial g_{n,2}}{\partial z} + \frac{\partial g_{n,2}}{\partial g_{n+1,2}}\frac{\partial g_{n+1,2}}{\partial z} \\ &= \sum_{j=n}^{n+1}\left(\prod_{m=n+1}^{j}\frac{\partial g_{m-1,2}}{\partial g_{m,2}}\right)\frac{\partial g_{j,2}}{\partial z},\end{aligned}$$

where $\prod_{m=n+1}^{n}\cdots = 1$. We now assume that Eq. (5) holds for $N = n+k$, i.e.

$$P_k : \frac{\partial f_{n\to N,2}}{\partial z} = \sum_{j=n}^{n+k}\left(\prod_{m=n+1}^{j}\frac{\partial g_{m-1,2}}{\partial g_{m,2}}\right)\frac{\partial g_{j,2}}{\partial z}.$$

To show that this implies P_{k+1}, we first write $f_{n\to N}(z,\zeta) = g_n(f_{n+1\to N}(z,\zeta))$. Now with a simple change in indexing of P_k we have

$$\frac{\partial f_{n+1\to N,2}}{\partial z} = \sum_{j=n+1}^{n+k+1}\left(\prod_{m=n+2}^{j}\frac{\partial g_{m-1,2}}{\partial g_{m,2}}\right)\frac{\partial g_{j,2}}{\partial z},$$

so that

$$\begin{aligned}P_{k+1} : \frac{\partial f_{n\to N,2}}{\partial z} &= \frac{\partial g_{n,2}}{\partial f_{n+1\to N,1}}\frac{\partial f_{n+1\to N,1}}{\partial z} + \frac{\partial g_{n,2}}{\partial f_{n+1\to N,2}}\frac{\partial f_{n+1\to N,2}}{\partial z} \\ &= \frac{\partial g_{n,2}}{\partial z} + \frac{\partial g_{n,2}}{\partial g_{n+1,2}}\sum_{j=n+1}^{n+k+1}\left(\prod_{m=n+2}^{j}\frac{\partial g_{m-1,2}}{\partial g_{m,2}}\right)\frac{\partial g_{j,2}}{\partial z} \\ &= \sum_{j=n}^{n+k+1}\left(\prod_{m=n+1}^{j}\frac{\partial g_{m-1,2}}{\partial g_{m,2}}\right)\frac{\partial g_{j,2}}{\partial z},\end{aligned}$$

which completes the proof.

We can now present our main result:

THEOREM 2. Under the assumptions of Theorem 1, $\mathrm{d}\,\mathrm{K}_{k=n}^{N}(a_k/b_k)/\mathrm{d}z$ is an analytic function for all z, $(z,0)\in G$, and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\mathop{\mathrm{K}}_{k=n}^{N}\left(\frac{a_k}{b_k}\right) &= \sum_{j=n}^{N}(-1)^{j-n+1}\left\{\prod_{k=n}^{j}\frac{1}{a_k}\left[\mathop{\mathrm{K}}_{\ell=k}^{N}\left(\frac{a_\ell}{b_\ell}\right)\right]^2\right\} \\ &\quad\times\left\{\left[\mathop{\mathrm{K}}_{\ell=j}^{N}\left(\frac{a_\ell}{b_\ell}\right)\right]^{-1}\frac{\mathrm{d}a_j}{\mathrm{d}z} - \frac{\mathrm{d}b_j}{\mathrm{d}z}\right\}. \qquad (6)\end{aligned}$$

PROOF. Applying substitutions

$$\frac{\partial g_{k,2}}{\partial z} = \left. \frac{\frac{\mathrm{d}a_k}{\mathrm{d}z}\,(b_k+\zeta) - a_k \frac{\mathrm{d}b_k}{\mathrm{d}z}}{(b_k+\zeta)^2} \right|_{\zeta=\zeta_k}$$

and

$$\frac{\partial g_{k,2}}{\partial \zeta} = -\left. \frac{a_k}{(b_k+\zeta)^2} \right|_{\zeta=\zeta_k}$$

with $\zeta_k = f_{k+1\to N,2} = \mathrm{K}_{m=k+1}^{N}(a_m/b_m)$ for $k = n, \ldots, N-1$; $\zeta_N = 0$, to the previous Lemma gives for the derivative

$$\sum_{j=n}^{N} \left\{ \prod_{k=n+1}^{j} \frac{-a_{k-1}}{\left[b_{k-1} + \mathop{\mathrm{K}}_{\ell=k}^{N} \left(\frac{a_\ell}{b_\ell} \right) \right]^2} \right\} \frac{\left[b_j + \mathop{\mathrm{K}}_{\ell=j+1}^{N} \left(\frac{a_\ell}{b_\ell} \right) \right] \frac{\mathrm{d}a_j}{\mathrm{d}z} - a_j \frac{\mathrm{d}b_j}{\mathrm{d}z}}{\left[b_j + \mathop{\mathrm{K}}_{\ell=j+1}^{N} \left(\frac{a_\ell}{b_\ell} \right) \right]^2},$$

from which we obtain the final result [Eq. (6)] after factoring out continued fractions. We can now replace the partial derivative with a total one as $\mathrm{K}_{k=n}^{N}(a_k/b_k)$ is an analytic function of z only.

It should be noted that if a_k and b_k are unknown analytic functions, conditions in Theorem 1 are sufficient to guarantee that the undetermined cases $0/0$ or $0 \cdot \infty$ are not possible to occur when evaluating $\mathrm{d}\,\mathrm{K}_{k=n}^{N}(a_k/b_k)/\mathrm{d}z$. However, if a_k and b_k are known, it might be possible to relax these conditions: for example, if either $a_\ell \equiv 0$ or $b_{\ell+1} \equiv -\mathrm{K}_{k=\ell+2}^{N}(a_k/b_k)$ for some $N-1 > \ell > n$, the original continued fraction simply terminates after $\ell - n$ levels and we can reset $N = \ell$.

3 Partial Derivatives with Respect to a_ℓ and b_ℓ

As an example of the application of Eq. (6), we turn our attention to two special cases presented in earlier literature which give partial derivatives of continued fractions with respect to their elements. In accordance with the standard practice in the literature, we consider only $n = 1$.

The *Nth modified approximant* of the continued fraction (1) can be now defined in our notation as $f_{1\to N,2}(z,\zeta)$ for arbitrary ζ [1, Sec. I.5]. This allows us to extend preceding results for infinite continued fractions:

OBSERVATION. The previous Lemma holds for all $\zeta \in B(0, R_G)$ that are independent of z as well as for $\zeta = 0$. Now if we consider sequences $\{a_k\}_{k\geq 1}$, $a_k \neq 0$ for $k < N$, and $\{b_k\}_{k\geq 1}$, which are constants except for subsequences $\{a_\ell(z)\}_{\ell\in I}$ and $\{b_\ell(z)\}_{\ell\in J}$, $I, J \subseteq \{1, 2, \ldots, N\}$ for some N, which are analytic functions of z so that

$\mathrm{K}_{k=N+1}^{\infty}(a_k/b_k)$ is defined and converges into $\zeta_\infty \in \hat{\mathbb{C}}$, the Theorem 2 also holds for $\zeta = \zeta_\infty$, i.e. for $\mathrm{K}_{k=1}^{\infty}(a_k/b_k)$.

The rationale behind this observation is that, as long as the $(\ell+1)$st tail, $\ell \le N$, is constant, the Nth modified approximant of (1) given by the finite composition $f_{1\to N}(z,\zeta_\infty)$ results $f_{1\to N,2}(z,\zeta_\infty) = \mathrm{K}_{k=1}^{\infty}(a_k/b_k)$. For the extension to an infinite composition, see [5].

Let us first consider the case where all b_k and $a_{k\neq\ell}$ are constants. Assuming that neither a_ℓ nor $\mathrm{d}a_\ell/\mathrm{d}z$ vanishes, only the term with index ℓ remains from the sum, and applying $\mathrm{d}z = (\partial a_\ell/\partial z)^{-1}\mathrm{d}a_\ell$ to Eq. (6) gives

$$
\begin{aligned}
\frac{\partial}{\partial a_\ell} \mathop{\mathrm{K}}_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right) &= (-1)^\ell \prod_{k=1}^{\ell} \frac{1}{a_k}\left[\mathop{\mathrm{K}}_{j=k}^{\infty}\left(\frac{a_j}{b_j}\right)\right]^2 \left[\mathop{\mathrm{K}}_{j=\ell}^{\infty}\left(\frac{a_j}{b_j}\right)\right]^{-1} \\
&= (-1)^{\ell-2} \prod_{k=2}^{\ell-1} \frac{1}{a_k}\left[\mathop{\mathrm{K}}_{j=k}^{\infty}\left(\frac{a_j}{b_j}\right)\right]^2 \frac{1}{a_1}\left[\mathop{\mathrm{K}}_{j=1}^{\infty}\left(\frac{a_j}{b_j}\right)\right]^2 \frac{1}{a_\ell}\mathop{\mathrm{K}}_{j=\ell}^{\infty}\left(\frac{a_j}{b_j}\right) \\
&= \frac{1}{a_\ell}\mathop{\mathrm{K}}_{j=1}^{\infty}\left(\frac{a_j}{b_j}\right) \prod_{k=2}^{\ell-1} \frac{-\mathop{\mathrm{K}}_{j=k}^{\infty}\left(\frac{a_j}{b_j}\right)}{b_k + \mathop{\mathrm{K}}_{j=k+1}^{\infty}\left(\frac{a_j}{b_j}\right)} \frac{\mathop{\mathrm{K}}_{j=\ell}^{\infty}\left(\frac{a_j}{b_j}\right)}{b_1 + \mathop{\mathrm{K}}_{j=2}^{\infty}\left(\frac{a_j}{b_j}\right)} \\
&= \frac{1}{a_\ell}\mathop{\mathrm{K}}_{j=1}^{\infty}\left(\frac{a_j}{b_j}\right) \prod_{k=2}^{\ell} \frac{-\mathop{\mathrm{K}}_{j=k}^{\infty}\left(\frac{a_j}{b_j}\right)}{b_{k-1} + \mathop{\mathrm{K}}_{j=k}^{\infty}\left(\frac{a_j}{b_j}\right)},
\end{aligned} \tag{7}
$$

which is the generalization of Waadeland's [6] formula by Levrie and Bultheel [7]; the original formula follows if $b_k \equiv 1$ and $\ell \ge 2$. Note that if $\ell = 1$, the minus sign cannot be taken inside the (empty) product.

Next we turn to the opposite case, and set all a_k and $b_{k\neq\ell}$ constants and assume that neither b_ℓ nor $\mathrm{d}b_\ell/\mathrm{d}z$ vanishes. Considering first the finite case, Eq. (6) reduces to

$$
\frac{\partial}{\partial b_\ell} \mathop{\mathrm{K}}_{k=1}^{N}\left(\frac{a_k}{b_k}\right) = (-1)^\ell \prod_{k=1}^{\ell} \frac{1}{a_k}\left[\mathop{\mathrm{K}}_{j=k}^{N}\left(\frac{a_j}{b_j}\right)\right]^2. \tag{8}
$$

Using the determinant formula [1, Eq. (1.2.10)], we can write

$$
(-1)^{\ell-1}\prod_{k=1}^{\ell} a_k = B_\ell B_{\ell-1}\left[\mathop{\mathrm{K}}_{k=1}^{\ell}\left(\frac{a_k}{b_k}\right) - \mathop{\mathrm{K}}_{k=1}^{\ell-1}\left(\frac{a_k}{b_k}\right)\right], \tag{9}
$$

where the mth canonical denominator B_m is given by the Wallis–Euler recurrence relation $B_m = b_m B_{m-1} + a_m B_{m-2}$ with initial conditions $B_{-1} = 0$ and $B_0 = 1$ and satisfies the relation $A_m/B_m = \mathrm{K}_{k=1}^{m}(a_k/b_k)$. On the other hand, Dudley [8, Corollary 1.10 that also holds in the complex case] has shown that

$$\frac{\partial}{\partial b_\ell} \mathop{\mathrm{K}}_{k=1}^{N}\left(\frac{a_k}{b_k}\right) = -\frac{\left[\mathop{\mathrm{K}}_{k=1}^{\ell-1}\left(\frac{a_k}{b_k}\right) - \mathop{\mathrm{K}}_{k=1}^{N}\left(\frac{a_k}{b_k}\right)\right]^2 B_{\ell-1}}{\left[\mathop{\mathrm{K}}_{k=1}^{\ell}\left(\frac{a_k}{b_k}\right) - \mathop{\mathrm{K}}_{k=1}^{\ell-1}\left(\frac{a_k}{b_k}\right)\right] B_\ell}. \tag{10}$$

Together Eqs. (8)–(10) give

$$\prod_{k=1}^{\ell}\left[\mathop{\mathrm{K}}_{j=k}^{N}\left(\frac{a_j}{b_j}\right)\right]^2 = B_{\ell-1}^2\left[\mathop{\mathrm{K}}_{k=1}^{N}\left(\frac{a_k}{b_k}\right) - \mathop{\mathrm{K}}_{k=1}^{\ell-1}\left(\frac{a_k}{b_k}\right)\right]^2, \tag{11}$$

which seems to be a new result, though it can probably be obtained independently from the basic properties of continued fractions. Using the Observation above and Theorem 1.11 in [8], we can extend the previous result:

CONJECTURE. If $\mathrm{K}_{k=1}^{\infty}(a_k/b_k)$ converges in $\hat{\mathbb{C}}$,

$$\prod_{k=1}^{\ell}\left[\mathop{\mathrm{K}}_{j=k}^{\infty}\left(\frac{a_j}{b_j}\right)\right]^2 = B_{\ell-1}^2\left[\mathop{\mathrm{K}}_{k=1}^{\infty}\left(\frac{a_k}{b_k}\right) - \mathop{\mathrm{K}}_{k=1}^{\ell-1}\left(\frac{a_k}{b_k}\right)\right]^2. \tag{12}$$

We remark that although the derivation leading to Eqs. (11) and (12) was based on the assumption of analytic sequences $\{a_k\}_{k\geq 1}$ and $\{b_k\}_{k\geq 1}$, this is not necessary for the proposed Conjecture to hold, as we can always expand $\mathrm{K}_{k=1}^{\infty}(a_k/b_k)$ into series with elements that are rational functions of a_k and b_k [3, Sec. 1.7], and are therefore analytic with respect to a_ℓ or b_ℓ for all $\ell < \infty$.

4 Concluding Remarks

Despite its simplicity, the main result of this short note, Theorem 2, has not apparently been published before: As only intermediate complex analysis is needed to prove the result, it might have some instructional use besides being relevant for scientists working in adjacent fields. It should be noted that in practical applications, a_k and b_k are typically relatively simple functions, such as low-degree polynomials or rational functions, so that the relevant domain for the problem D, $D \times B(0, R_G) \subseteq G$, can be often inferred directly from the context.

The fact that the obtained formula contains the original continued fraction and its tails yields some useful corollaries: First, for relatively simple a_k and b_k the computational cost of numerical evaluation of the analytic expression [Eq. (6)] for the derivative is not much higher than that of the original continued fraction if a vector containing computed tails/approximants is updated during each iteration step, and can be lower than that of the evaluation of the corresponding finite difference that involves two evaluations of $\mathrm{K}_{k=n}^{N}(a_k/b_k)$ at different points. Secondly, the obtained result can be applied iteratively to calculate higher order derivatives of the finite continued fraction if needed. However, expressions for these higher derivatives become increasingly impractical with increasing order, and we make no attempt to present those here; we are unaware of any suitable generalization of the Faá di Bruno's formula that could simplify this treatment.

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