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The Higher Derivatives Of The Inverse Tangent Function Revisited*

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Abstract

A closed-form formula for all derivatives of the real arctangent function is presented. In addition a curious series expansion for the function is obtained and one of its specific consequences is given.

1 Introduction

Constructing Maclaurin series expansion for the arctan function is easy by using an integral. However, Taylor series expansion around an arbitrary point is not so simple.

It can be easily verified, by induction, that the function $\arctan x$ possesses on $\mathbb{R}$ derivatives of all orders. More precisely, there exists the sequence $\left(P_n\right)_{n\in\mathbb{N}}$ of polynomials such that

$$\frac{d^n}{dx^n}(\arctan x) \equiv \frac{P_{n-1}(x)}{(1+x^2)^n}$$

and the degree of $P_n(x)$ does not exceed n. Obviously, these polynomials satisfy the recursion relation $P_n(x) \equiv (1+x^2)P'_{n-1}(x) - 2nxP_{n-1}(x)$ with $P_0(x) \equiv 1$. To our knowledge the closed form formula for $P_n(x)$ is still unknown.

On the contrary, many different ways of how to find consecutive derivatives of arctan at $x=0$ are known, besides the method mentioned above. One of them, for example, is the iterative method. Namely, the function $y(x) \equiv \arctan x$ has the derivative $y'(x) \equiv \frac{1}{1+x^2}$; consequently, the identity $(1+x^2)y'(x) \equiv 1$ holds true. Hence, using Leibniz rule for the n-th derivative we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (1+x^2)^{(k)} \left(y^{(1)}\right)^{(n-k)} \equiv 0,$$

that is

$$(1+x^2)\cdot y^{(n+1)}(x) + 2nx\cdot y^{(n)}(x) + n(n-1)\cdot y^{(n-1)}(x) \equiv 0, \tag{1}$$

for $x\in\mathbb{R}$ and $n\geq 1$. Thus, we get the recursion

$$y^{(n+1)}(0) = -n(n-1)\cdot y^{(n-1)}(0), \quad n\geq 1,$$

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which results in

$$y^{(2k)}(0) = 0 \qquad \text{and} \qquad y^{(2k+1)}(0) = (-1)^k(2k!), \quad k \geq 0. \tag{2}$$

To generate Taylor series expansion around an arbitrary x directly we need the higher derivatives at this point. However, to find $y^{(n)}(x)$ from (1) it is not easy. In [1] the authors used a brilliant idea how to calculate it. Unfortunately, they were not very careful in their analysis and made some errors in their derivations and in the final results as well. The fact that the Theorem 1 in [1] is not valid is evident from the observation that the derivatives of arctan function of even orders are odd functions[1]. However, the functions R_n, $R_n(x)$ being the right hand side of the Eq. (1) in Theorem 1 [1], are even for every n. Figures 1a and 1b, using [3], show the graphs of the derivatives $\arctan^{(6)}(x) \equiv 240x(-3+10x^2-3x^4)(1+x^2)^{-6}$ and $\arctan^{(8)}(x) \equiv 40320x(1-7x^2+7x^4-x^6)(1+x^2)^{-8}$, together with the graphs of the functions $R_6(x)$ and $R_8(x)$ (thick, dashed lines). We have the coincidence on $\mathbb{R}^+$, but not on $\mathbb{R}^-$.

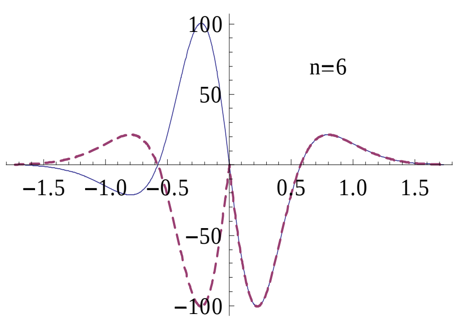

Figure 1a: The graphs of the functions $\arctan^{(6)}(x)$ and $R_6(x)$

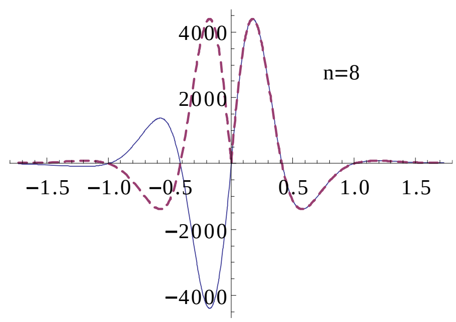

Figure 1b: The graphs of the functions $\arctan^{(8)}(x)$ and $R_8(x)$

Similarly, the sum, and also all partial sums, of the series in [1, Theorem 2, Eq. (6)] are even functions, but arctan is an odd one. Figure 2 shows, using [3], the graph of the function arctan together with the graph of 500-th partial sum $S_{500}(x)$ of the series on the right of Eq. (6) [1] (thick, dashed line). The graphs coincide on $\mathbb{R}^+$, but evidently not on $\mathbb{R}^-$.

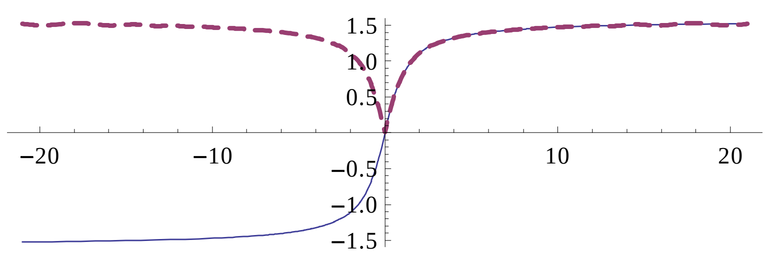

Figure 2: The graphs of $\arctan(x)$ and $S_{500}(x)$

[1] generally: f odd (even) $\Longrightarrow f'$ even (odd)

We wish to improve the contribution [1] by giving the correct derivations and correct results.

2 Higher Derivatives

We shall show how the authors's idea can be used successfully. To this effect we reformulate the Theorem 1 in its correct version which differs from the original one by the inclusion of factor $\mathrm{sg}^{n-1}(x)$ where $\mathrm{sg}(x)$ is defined as

$$\mathrm{sg}(x) := \begin{cases} -1, & x < 0 \\ \ \ 1, & x \geq 0\ . \end{cases}$$

Hence $\mathrm{sg}(x)$ is different from zero everywhere, $x \equiv \mathrm{sg}(x) \cdot |x|$, $1/\mathrm{sg}(x) \equiv \mathrm{sg}(x)$ and $\mathrm{sg}(-x) = -\mathrm{sg}(x)$ for $x \in \mathbb{R} \setminus \{0\}$.

THEOREM 1. For $x \in \mathbb{R}$ and $n \geq 1$ there holds the equality

$$\frac{d^n}{dx^n}(\arctan x) = \mathrm{sg}^{n-1}(-x) \cdot \frac{(n-1)!}{(1+x^2)^{n/2}} \cdot \sin\left(n \cdot \arcsin \frac{1}{\sqrt{1+x^2}}\right). \tag{3}$$

PROOF. The equality (3) is obviously true for $n = 1$ and any real x since in this case the right-hand side of the equation (3) becomes equal to

$$\mathrm{sg}^0(-x) \cdot 0! \cdot \frac{1}{\sqrt{1+x^2}} \cdot \frac{1}{\sqrt{1+x^2}} \equiv \frac{1}{1+x^2}\,.$$

Moreover, according to (2), the relation (3) is true also for $x = 0$ and $n \geq 1$ because the right-hand side of the equation (3) then becomes equal to

$$\mathrm{sg}^{n-1}(0) \cdot (n-1)! \cdot \sin\left(n\frac{\pi}{2}\right) = \begin{cases} (-1)^{(n-1)/2}(n-1)!\,, & n \text{ odd} \\ \qquad 0\,, & n \text{ even}\,. \end{cases}$$

Now we have to show that the identity (3) is valid also for $x \in \mathbb{R} \setminus \{0\}$ and $n > 1$. To do this we introduce the auxiliary function $\varphi\colon \mathbb{R} \to \mathbb{R}$,

$$\varphi(x) := \arcsin \frac{1}{\sqrt{1+x^2}} \in \left(0, \frac{\pi}{2}\right]\,, \tag{4}$$

being continuous and differentiable on $\mathbb{R}^- \cup \mathbb{R}^+$ with the derivative

$$\varphi'(x) = -\frac{x}{|x|} \cdot \frac{1}{1+x^2} = \begin{cases} \ \ \frac{1}{1+x^2}\,, & x < 0 \\ -\frac{1}{1+x^2}\,, & x > 0\,. \end{cases} \tag{5}$$

Referring to (4) we have

$$\sin\big(\varphi(x)\big) \equiv \frac{1}{\sqrt{1+x^2}} \quad x \in \mathbb{R}. \tag{6}$$

Consequently, there holds the equality

$$\varphi'(x) = \begin{cases} \sin^2\big(\varphi(x)\big)\,, & x<0 \qquad \cdots \quad (*) \\ -\sin^2\big(\varphi(x)\big)\,, & x>0 \qquad \cdots \quad (**)\,. \end{cases} \tag{7}$$

Remark. Contrary to the supposition that was probably made by the authors [1, p. 71], the function $\varphi(x)$ is not differentiable at $x=0$. As a matter of fact, at this point it has finite left and right derivatives which are, unfortunately, different. Indeed, using L'Hôpital rule we have

$$\begin{aligned} {\varphi'}_{-(+)}(0) &= \lim_{h\uparrow 0\,(h\downarrow 0)} \frac{1}{h}\left(\arcsin\frac{1}{\sqrt{1+h^2}} - \frac{\pi}{2}\right) \\ &= -\lim_{h\uparrow 0\,(h\downarrow 0)} \frac{h}{|h|(1+h^2)} = +1\,(=-1)\,. \end{aligned}$$

The graph of the function $\varphi(x)$ is depicted, using [3], in Figure 3.

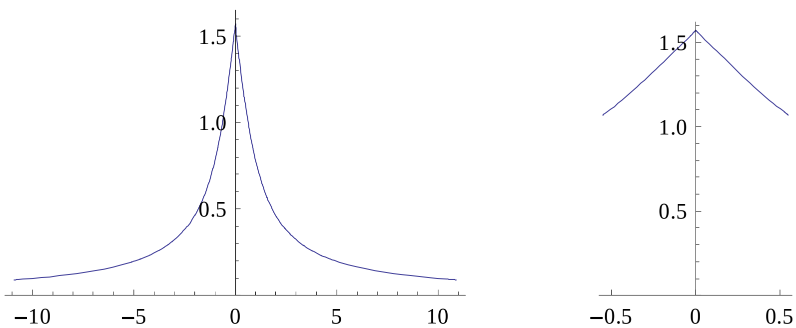

Figure 3: The graph of the function $\varphi(x)$

Using (4) and (6), the equation (3) is transformed into the following equivalent identity, for $x\in\mathbb{R}$,

$$\frac{d^n}{dx^n}(\arctan x) = \mathrm{sg}^{n-1}(-x)\cdot(n-1)!\cdot\sin^n\big(\varphi(x)\big)\cdot\sin\Big(n\sin\big(\varphi(x)\big)\Big)\,. \tag{8}$$

For $x\in\mathbb{R}^+$ the relation (8) reduces to the equality

$$\frac{d^n}{x^n}(\arctan x) = (-1)^{n-1}\cdot(n-1)!\cdot\sin^n\big(\varphi(x)\big)\cdot\sin\Big(n\sin\big(\varphi(x)\big)\Big)\,,$$

which, by induction, could be easily verified [1, p. 71] using (7). Hence (3) holds true for $x>0$ and $n\geq 1$.

Consequently, for $x\in\mathbb{R}^-$ the relation (5) is also valid since in this case, substituting $x=-t$ with $t=t(x)=|x|=-x$, we have

$$\frac{d^n}{dx^n}(\arctan x) = \frac{d^n}{dx^n}\arctan\big(-t(x)\big) = \left[\frac{d^n}{dt^n}(-\arctan t)\right]_{t=-x}\cdot\left(\frac{dt}{dx}(x)\right)^n$$

$$
\begin{aligned}
&= -\left[\frac{d^n}{dt^n}\big(\arctan t\big)\right]_{t=-x} \cdot (-1)^n \\
&= -\operatorname{sg}^{n-1}(+x) \cdot \frac{(n-1)!}{(1+x^2)^{n/2}} \cdot \sin\left(n \cdot \arcsin \frac{1}{\sqrt{1+x^2}}\right) \cdot (-1)^n \\
&= \operatorname{sg}^{n-1}(-x) \cdot \frac{(n-1)!}{(1+x^2)^{n/2}} \cdot \sin\left(n \cdot \arcsin \frac{1}{\sqrt{1+x^2}}\right).
\end{aligned}
$$

3 Curious Series Expansion

The function f given as complex curvilinear integral,

$$
f(z) := \int_0^z \frac{d\zeta}{1+\zeta^2}\,,
$$

is analytic on the cut complex plane, i.e. in the domain $\mathcal{D} = \mathbb{C} \setminus \{z \in \mathbb{C} \,|\, \mathrm{Re} z = 0,\ |\mathrm{Im} z| \geq 1\}$ and there it has the complex derivative $f'(z) = 1/(1+z^2)$ [2, Th. 13.5, p. 282]. Particularly, we have

$$
f'(x) = \frac{1}{1+x^2} = \arctan'(x), \quad x \in \mathbb{R}.
$$

Hence, $f(z) = \arctan z$ for $z \in \mathbb{R}$; f is an analytic continuation of real function arctan. Due to its analyticity, f can be expanded into Taylor's series around every $z_0 \in \mathcal{D}$ and the obtained power series is convergent on every open disk centered at z_0 and included in $\mathcal{D}$ [2, Th. 16.7, p. 361].

For any $x \in \mathbb{R}$ the number $x + (-x) = 0$ belongs to the open disk $|z - x| < |x \mp \mathrm{i}|$, which is included in $\mathcal{D}$. Therefore, on this disk f can be expanded into Taylor's series; consequently

$$
0 = f\big(x + (-x)\big) = f(x) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x)}{n!}\,(-x)^n,
$$

and we get the expansion

$$
\boxed{\arctan x = -\sum_{n=1}^{\infty} \frac{\arctan^{(n)}(x)}{n!}\,(-x)^n,} \tag{9}
$$

true for every $x \in \mathbb{R}$.

Now, from (9) and (3) we obtain the following expansions

$$
\begin{aligned}
\arctan x &= -\sum_{n=1}^{\infty} \frac{1}{n!} \cdot \operatorname{sg}^{n-1}(-x) \cdot \frac{(n-1)!}{(1+x^2)^{n/2}} \cdot \sin\left(n \cdot \arcsin \frac{1}{\sqrt{1+x^2}}\right) \cdot (-x)^n \\
&= -\operatorname{sg}(-x) \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{\big(-x \operatorname{sg}(-x)\big)^n}{(1+x^2)^{n/2}} \cdot \sin\left(n \cdot \arcsin \frac{1}{\sqrt{1+x^2}}\right)
\end{aligned}
$$

$$= \operatorname{sg}(x) \sum_{n=1}^{\infty} \frac{|-x|^n}{n(1+x^2)^{n/2}} \cdot \sin\left(n \cdot \arcsin \frac{1}{\sqrt{1+x^2}}\right).$$

Thus, we arrive at the following theorem.

THEOREM 2. For any $x \in \mathbb{R}$ there holds the equality

$$\arctan x = \operatorname{sg}(x) \cdot \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x^2}{1+x^2}\right)^{n/2} \sin\left(n \cdot \arcsin \frac{1}{\sqrt{1+x^2}}\right). \tag{10}$$

In Figure 4 are depicted the graph of the function arctan and the graph (dashed line) of the 100-th partial sum of the series in the right hand side of the equation (10).

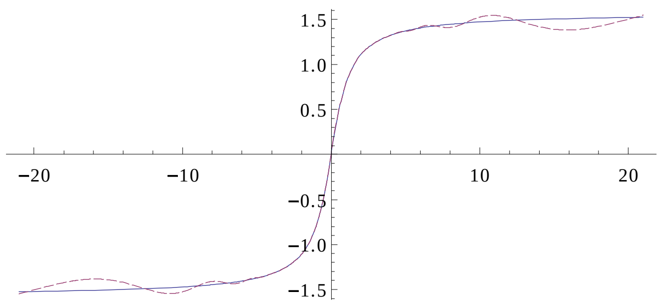

Figure 4: The graph of $\arctan(x)$ and its series approximation using the 100-partial sum in (10)

4 π-Series

The immediate consequence of Theorem 2 is the following result.

THEOREM 3. For $\varphi \in \mathbb{R}$ such that $0 < |\varphi| < \pi$, and only for such φ, there holds the equality

$$\frac{\pi}{2} - |\varphi| = \operatorname{sg}(\varphi) \cdot \sum_{n=1}^{\infty} \frac{1}{n} (\cos\varphi)^n \sin(n\,\varphi). \tag{11}$$

PROOF. A) $\boxed{0 < \varphi < \frac{\pi}{2} :}$ In this case we consider the variable

$$x := \sqrt{\sin^{-2}(\varphi) - 1} > 0.$$

We obtain

$$\varphi = \arcsin \frac{1}{\sqrt{1+x^2}} \quad \text{and} \quad \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2} = 1 - \sin^2\varphi = \cos^2\varphi \tag{12}$$

and

$$\cot^2\varphi = \frac{1}{\sin^2(\varphi)} - 1 = \left(1+x^2\right) - 1 = x^2.$$

Consequently, since $\varphi \in \left(0, \frac{\pi}{2}\right)$, it follows that $\tan\left(\frac{\pi}{2} - \varphi\right) = \cot\varphi = x$. Hence,

$$\frac{\pi}{2} - \varphi = \arctan x. \tag{13}$$

Under given suppositions, the relations (13), (10) and (12) confirm the identity (11).

B) $\boxed{-\frac{\pi}{2} < \varphi < 0 :}$ Under this condition we estimate $0 < -\varphi < \frac{\pi}{2}$. Consequently, considering the preceding result, we have

$$\frac{\pi}{2} - (-\varphi) = \sum_{n=1}^{\infty} \frac{1}{n} \big(\cos(-\varphi)\big)^n \sin(-n\,\varphi),$$

that is

$$\frac{\pi}{2} - |\varphi| = -\sum_{n=1}^{\infty} \frac{1}{n} (\cos\varphi)^n \sin(n\,\varphi).$$

Thus, the validity of the relation (11) is confirmed once again.

C) $\boxed{-\pi < \varphi < -\frac{\pi}{2} :}$ In this case the estimate $0 < \varphi + \pi < \frac{\pi}{2}$ holds. Therefore, using the first result, we obtain

$$\frac{\pi}{2} - (\varphi + \pi) = \sum_{n=1}^{\infty} \frac{1}{n} \big(-\cos\varphi\big)^n (-1)^n \sin(n\,\varphi),$$

that is

$$\frac{\pi}{2} - |\varphi| = -\sum_{n=1}^{\infty} \frac{1}{n} \cos^n(\varphi) \sin(n\,\varphi)$$

and (11) is approved repeatedly.

D) $\boxed{\frac{\pi}{2} < \varphi < \pi :}$ Under this condition we have $0 < \pi - \varphi < \frac{\pi}{2}$. Thus, referring to the first result, we have

$$\begin{aligned}\frac{\pi}{2} - (\pi - \varphi) &= \sum_{n=1}^{\infty} \frac{1}{n} \big(-\cos\varphi\big)^n (-1)^{n+1} \sin(n\,\varphi) \\ &= -\sum_{n=1}^{\infty} \frac{1}{n} \big(\cos\varphi\big)^n \sin(n\,\varphi)\end{aligned}$$

that is

$$\frac{\pi}{2} - \varphi = \sum_{n=1}^{\infty} \frac{1}{n} \cos^n(\varphi) \sin(n\,\varphi)$$

and (11) is verified also in this last case.

The function F, $F(\varphi) := \mathrm{sg}(\varphi) \cdot \sum_{n=1}^{\infty} \frac{1}{n} (\cos\varphi)^n \sin(n\,\varphi)$, fulfill the identities $F(\varphi + 2\pi) \equiv F(\varphi)$, for $\varphi > 0$, and $F(\varphi - 2\pi) \equiv F(\varphi)$, for $\varphi < 0$. Hence, the equality (11) cannot be true for $\varphi \in \mathbb{R} \setminus [-\pi, \pi]$.

Figure 5 illustrates the relation (11) by plotting, for $\varphi \in \bigcup_{k=0}^{5} \big((-3+k)\pi+0.013,\ (-2+k)\pi-0.013\big)$, the graph of the function $\varphi \mapsto \frac{\pi}{2} - |\varphi|$ (dashed line) and the graph of the function $\varphi \mapsto \mathrm{sg}(\varphi) \cdot \sum_{n=1}^{100} \frac{1}{n} (\cos\varphi)^n \sin(n\,\varphi)$.

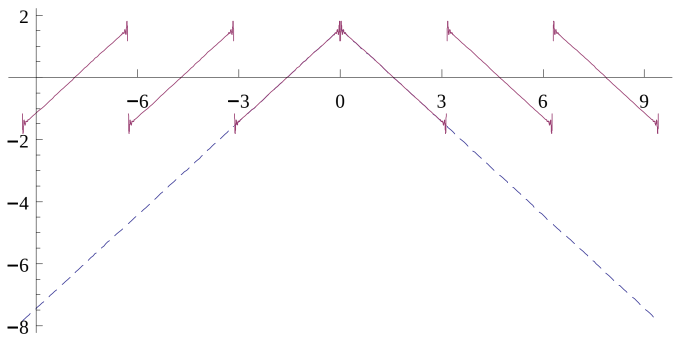

Figure 5: The graph of the function $\varphi \mapsto \frac{\pi}{2} - |\varphi|$ (dashed line) and its series approximation using the 100-th partial sum

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