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Inequalities For Spreads Of Matrix Sums And Products*

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Abstract

Let $\mathbf{A}$ and $\mathbf{B}$ be complex matrices of same dimension. Given their eigenvalues and singular values, we survey and further develop simple inequalities for eigenvalues and singular values of $\mathbf{A}+\mathbf{B}$, $\mathbf{AB}$, and $\mathbf{A}\circ\mathbf{B}$. Here $\circ$ denotes the Hadamard product. As corollaries, we find inequalities for additive and multiplicative spreads of these matrices.

1 Introduction

Let $\mathbf{A}$ be a complex $n\times n$ matrix (assume $n\geq 2$ throughout) with eigenvalues $\lambda_1,...,\lambda_n$, denoted also by $\lambda_i(\mathbf{A})=\lambda_i$. Order them $\lambda_1\geq ...\geq\lambda_n$ if they are real. In the general case, order them in absolute value: $|\lambda_{(1)}|\geq ...\geq|\lambda_{(n)}|$, and denote also $|\lambda_{(i)}(\mathbf{A})|=|\lambda_{(i)}|$. We define the *additive spread* of $\mathbf{A}$ by

$$\text{ads } \mathbf{A}=\max_{i,j}|\lambda_i-\lambda_j|$$

and *multiplicative spread* (assuming the λ_i's nonzero) by

$$\text{mls } \mathbf{A}=\max_{i,j}\left|\frac{\lambda_i}{\lambda_j}\right|.$$

Several inequalities for the additive spread are known (see [9] and its references). The multiplicative spread of a Hermitian positive definite matrix, the *Wielandt ratio*, is widely studied (see [1] and its references).

Let $\sigma_1\geq ...\geq\sigma_n(\geq 0)$ be the singular values of $\mathbf{A}$, denoted also by $\sigma_i(\mathbf{A})=\sigma_i$. If $\mathbf{A}$ is nonsingular (i.e., if $\sigma_n>0$) we define its *(spectral) condition number* by

$$\text{cnd}\,\mathbf{A}=(\text{mls}\,\mathbf{A}^*\mathbf{A})^{\frac{1}{2}}=\frac{\sigma_1}{\sigma_n}.$$

It measures the numerical instability of $\mathbf{A}$ (see e.g. [4]).

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Let $1 \leq i < j \leq n$.

If the eigenvalues of $\mathbf{A}$ are real, we define the *additive mid-spread* of $\mathbf{A}$ by

$$\operatorname{ads}_{ij} \mathbf{A} = \lambda_i - \lambda_j$$

and the *multiplicative mid-spread* by

$$\operatorname{mls}_{ij} \mathbf{A} = \frac{\lambda_i}{\lambda_j} \quad (\lambda_j > 0).$$

Assuming nothing about eigenvalues, we define the *absolute multiplicative spread* of $\mathbf{A}$ by

$$\operatorname{Mls} \mathbf{A} = \frac{|\lambda_{(1)}|}{|\lambda_{(n)}|} \qquad (\lambda_{(n)} \neq 0)$$

and the *absolute multiplicative mid-spread* by

$$\operatorname{Mls}_{ij} \mathbf{A} = \frac{|\lambda_{(i)}|}{|\lambda_{(j)}|} \quad (\lambda_{(j)} \neq 0).$$

We do not find the absolute additive spread interesting. Finally, we define the *mid-condition number* of $\mathbf{A}$ by

$$\operatorname{cnd}_{ij} \mathbf{A} = \frac{\sigma_i}{\sigma_j} \quad (\sigma_j > 0)$$

(although they may be bad measures of condition).

There are many well-known inequalities for eigenvalues and singular values of $\mathbf{A}+\mathbf{B}$, $\mathbf{AB}$, and $\mathbf{A} \circ \mathbf{B}$, when the eigenvalues and singular values of $\mathbf{A}$ and $\mathbf{B}$ are given, see e.g. [2], [3], [5], [6], [7], [8], [10], [11], [12]. Here $\circ$ denotes the Hadamard product. We will survey and further develop simple inequalities. We are particularly interested in their analogies. As corollaries, we will find inequalities for spreads of $\mathbf{A}+\mathbf{B}$, $\mathbf{AB}$, and $\mathbf{A} \circ \mathbf{B}$, when the spreads of $\mathbf{A}$ and $\mathbf{B}$ are given.

There is a deep theory behind the eigenvalues of the sum of Hermitian matrices and the singular values of the product of square matrices (see e.g. [3] and its references), but our approach is elementary.

2 Eigenvalues of $\mathbf{A} + \mathbf{B}$

If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, then we can both underestimate and overestimate eigenvalues of $\mathbf{A}+\mathbf{B}$ by using eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. Hence we can overestimate spreads of $\mathbf{A}+\mathbf{B}$ by using spreads of $\mathbf{A}$ and $\mathbf{B}$.

THEOREM 1 (Weyl, see e.g. [2], Theorem III.2.1; [6], Theorem 4.3.7). Let $\mathbf{A}$ and $\mathbf{B}$ be Hermitian $n \times n$ matrices. If $1 \leq k \leq i \leq n$ and $1 \leq l \leq n-i+1$, then

$$\lambda_{i+l-1}(\mathbf{A}) + \lambda_{n-l+1}(\mathbf{B}) \leq \lambda_i(\mathbf{A}+\mathbf{B}) \leq \lambda_{i-k+1}(\mathbf{A}) + \lambda_k(\mathbf{B}). \tag{1}$$

In particular,

$$\lambda_i(\mathbf{A}) + \lambda_n(\mathbf{B}) \leq \lambda_i(\mathbf{A}+\mathbf{B}) \leq \lambda_i(\mathbf{A}) + \lambda_1(\mathbf{B}) \tag{2}$$

and further

$$\lambda_n(\mathbf{A}) + \lambda_n(\mathbf{B}) \le \lambda_n(\mathbf{A}+\mathbf{B}),$$
$$\lambda_1(\mathbf{A}+\mathbf{B}) \le \lambda_1(\mathbf{A}) + \lambda_1(\mathbf{B}).$$

Lidskii's sum inequalities and their further developments (see e.g. [2], [3]) generalize (2).

COROLLARY 2. Let $\mathbf{A}$ and $\mathbf{B}$ be Hermitian $n \times n$ matrices. If $1 \le k \le i < j \le n$ and $1 \le l \le n-j+1$, then

$$\operatorname{ads}_{ij}(\mathbf{A}+\mathbf{B}) \le \operatorname{ads}_{i-k+1,j+l-1}\mathbf{A} + \operatorname{ads}_{k,n-l+1}\mathbf{B}. \tag{3}$$

In particular,

$$\operatorname{ads}_{ij}(\mathbf{A}+\mathbf{B}) \le \operatorname{ads}_{ij}\mathbf{A} + \operatorname{ads}\mathbf{B}$$

and further

$$\operatorname{ads}(\mathbf{A}+\mathbf{B}) \le \operatorname{ads}\mathbf{A} + \operatorname{ads}\mathbf{B}. \tag{4}$$

PROOF. In

$$\operatorname{ads}_{ij}(\mathbf{A}+\mathbf{B}) = \lambda_i(\mathbf{A}+\mathbf{B}) - \lambda_j(\mathbf{A}+\mathbf{B}),$$

apply the second inequality of (1) to the first term and the first inequality to the second. Then (3) follows.

COROLLARY 3. Let $\mathbf{A}$ and $\mathbf{B}$ be Hermitian $n \times n$ matrices and let i, j, k, l satisfy the conditions of Corollary 2. If $\lambda_{j+l-1}(\mathbf{A}) > 0$ and $\lambda_{n-l+1}(\mathbf{B}) > 0$, then

$$\operatorname{mls}_{ij}(\mathbf{A}+\mathbf{B}) < \operatorname{mls}_{i-k+1,j+l-1}\mathbf{A} + \operatorname{mls}_{k,n-l+1}\mathbf{B}. \tag{5}$$

In particular, if $\mathbf{B}$ is positive definite, then

$$\operatorname{mls}_{ij}(\mathbf{A}+\mathbf{B}) < \operatorname{mls}_{ij}\mathbf{A} + \operatorname{mls}\mathbf{B},$$

and if also $\mathbf{A}$ is positive definite, then

$$\operatorname{mls}(\mathbf{A}+\mathbf{B}) < \operatorname{mls}\mathbf{A} + \operatorname{mls}\mathbf{B}. \tag{6}$$

PROOF. Denoting $\alpha_p = \lambda_p(\mathbf{A})$, $\beta_p = \lambda_p(\mathbf{B})$ $(1 \le p \le n)$, $r = i-k+1$, $s = j+l-1$, $t = n-l+1$, we have by (1)

$$\operatorname{mls}_{ij}(\mathbf{A}+\mathbf{B}) = \frac{\lambda_i(\mathbf{A}+\mathbf{B})}{\lambda_j(\mathbf{A}+\mathbf{B})} \le \frac{\alpha_r + \beta_k}{\alpha_s + \beta_t}.$$

Since

$$\frac{\alpha_r}{\alpha_s} + \frac{\beta_k}{\beta_t} - \frac{\alpha_r + \beta_k}{\alpha_s + \beta_t} = \frac{\alpha_s^2 \beta_k + \alpha_r \beta_t^2}{\alpha_s \beta_t (\alpha_s + \beta_t)} > 0,$$

inequality (5) follows.

To study the additive spread of the sum of non-Hermitian matrices, we recall

THEOREM 4 ([9], Theorem 2 and Lemma 2). If $\mathbf{A}$ is a square matrix, then

$$\operatorname{ads}\mathbf{A} \le \max_{|z|=1} \operatorname{ads}\frac{z\mathbf{A}+\bar{z}\mathbf{A}^*}{2}. \tag{7}$$

If $\mathbf{A}$ is normal, then

$$\operatorname{ads}\mathbf{A} = \max_{|z|=1} \operatorname{ads}\frac{z\mathbf{A}+\bar{z}\mathbf{A}^*}{2}. \tag{8}$$

According to Corollary 2, inequality (4) holds for Hermitian matrices. We extend it to normal matrices.

THEOREM 5. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of same dimension, then

$$\operatorname{ads}(\mathbf{A}+\mathbf{B}) \le \max_{|z|=1} \operatorname{ads}\frac{z\mathbf{A}+\bar{z}\mathbf{A}^*}{2} + \max_{|z|=1} \operatorname{ads}\frac{z\mathbf{B}+\bar{z}\mathbf{B}^*}{2}. \tag{9}$$

If $\mathbf{A}$ and $\mathbf{B}$ are normal, then

$$\operatorname{ads}(\mathbf{A}+\mathbf{B}) \le \operatorname{ads}\mathbf{A} + \operatorname{ads}\mathbf{B}. \tag{10}$$

PROOF. By (7),

$$\operatorname{ads}(\mathbf{A}+\mathbf{B}) \le \max_{|z|=1} \operatorname{ads}\frac{z(\mathbf{A}+\mathbf{B})+\bar{z}(\mathbf{A}+\mathbf{B})^*}{2}.$$

Let z_0 be the maximizer. Recalling (4), we have

$$\begin{aligned}
\operatorname{ads}\frac{z_0(\mathbf{A}+\mathbf{B})+\bar{z}_0(\mathbf{A}+\mathbf{B})^*}{2} &= \operatorname{ads}\Big(\frac{z_0\mathbf{A}+\bar{z}_0\mathbf{A}^*}{2} + \frac{z_0\mathbf{B}+\bar{z}_0\mathbf{B}^*}{2}\Big) \\
&\le \operatorname{ads}\frac{z_0\mathbf{A}+\bar{z}_0\mathbf{A}^*}{2} + \operatorname{ads}\frac{z_0\mathbf{B}+\bar{z}_0\mathbf{B}^*}{2} \\
&\le \max_{|z|=1} \operatorname{ads}\frac{z\mathbf{A}+\bar{z}\mathbf{A}^*}{2} + \max_{|z|=1} \operatorname{ads}\frac{z\mathbf{B}+\bar{z}\mathbf{B}^*}{2},
\end{aligned}$$

which proves (9). Now (8) implies (10).

It is not sensible to ask whether (6) can be generalized for normal matrices, since it requires that the eigenvalues are real and positive. The counterexample

$$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}, \quad \mathbf{A}+\mathbf{B} = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$$

shows that (6) cannot be generalized for matrices with real and positive eigenvalues. We have $\operatorname{mls}(\mathbf{A}+\mathbf{B}) = 7$ but $\operatorname{mls}\mathbf{A} + \operatorname{mls}\mathbf{B} = 2$.

3 Singular Values of $\mathbf{A} + \mathbf{B}$

The following theorem is analogous to the second parts of (1) and (2).

THEOREM 6 (Fan, see e.g. [2], Problem III.6.5; [7], Theorem 3.3.16; [8], p. 243). Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $1 \leq k \leq i \leq n$, then

$$\sigma_i(\mathbf{A} + \mathbf{B}) \leq \sigma_{i-k+1}(\mathbf{A}) + \sigma_k(\mathbf{B}). \tag{11}$$

In particular,

$$\sigma_i(\mathbf{A} + \mathbf{B}) \leq \sigma_i(\mathbf{A}) + \sigma_1(\mathbf{B})$$

and further

$$\sigma_1(\mathbf{A} + \mathbf{B}) \leq \sigma_1(\mathbf{A}) + \sigma_1(\mathbf{B}).$$

The first parts of (1) and (2) do not have analogies for singular values. In other words: For $1 \leq l \leq n - i + 1$,

$$\sigma_{i+l-1}(\mathbf{A}) + \sigma_{n-l+1}(\mathbf{B}) \leq \sigma_i(\mathbf{A} + \mathbf{B})$$

is not valid in general. A counterexample is $\mathbf{A} = \mathbf{I}$, $\mathbf{B} = -\mathbf{I}$. For $i = l = 1$, this inequality "follows" from the wrong inequality 9.G.1.e (also 9.G.4.b) of [8].

There does not seem to be any good way to underestimate singular values of the sum by using singular values of the summands. Certainly (11) implies

$$|\sigma_{i+l-1}(\mathbf{A}) - \sigma_l(\mathbf{B})| \leq \sigma_i(\mathbf{A} + \mathbf{B}),$$

but this appears to be ineffective to our purpose.

Therefore we cannot apply our methods to the "additive singular value spread" $\sigma_1(\mathbf{A}) - \sigma_n(\mathbf{A})$.

4 Eigenvalues of $\mathbf{AB}$

If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian and nonnegative definite, then we can both underestimate and overestimate eigenvalues of $\mathbf{AB}$ by using eigenvalues of $\mathbf{A}$ and $\mathbf{B}$. Hence we can overestimate multiplicative spreads of $\mathbf{AB}$ by using those of $\mathbf{A}$ and $\mathbf{B}$. (The eigenvalues of $\mathbf{AB}$ are real, since

$$\lambda_i(\mathbf{AB}) = \lambda_i(\mathbf{A}^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}}\mathbf{B}) = \lambda_i(\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}),$$

and $\mathbf{A}^{\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}$ is Hermitian. The second equality follows from the fact that if $\mathbf{C}$ and $\mathbf{D}$ are square matrices of same order, then $\mathbf{CD}$ and $\mathbf{DC}$ have the same spectrum.)

Analogously to Theorem 1, we have

THEOREM 7. Let $\mathbf{A}$ and $\mathbf{B}$ be Hermitian nonnegative definite $n \times n$ matrices. If $1 \leq k \leq i \leq n$ and $1 \leq l \leq n - i + 1$, then

$$\lambda_{i+l-1}(\mathbf{A})\lambda_{n-l+1}(\mathbf{B}) \leq \lambda_i(\mathbf{AB}) \leq \lambda_{i-k+1}(\mathbf{A})\lambda_k(\mathbf{B}). \tag{12}$$

In particular,

$$\lambda_i(\mathbf{A})\lambda_n(\mathbf{B}) \le \lambda_i(\mathbf{AB}) \le \lambda_i(\mathbf{A})\lambda_1(\mathbf{B}) \tag{13}$$

and further

$$\lambda_n(\mathbf{A})\lambda_n(\mathbf{B}) \le \lambda_n(\mathbf{AB}), \qquad \lambda_1(\mathbf{AB}) \le \lambda_1(\mathbf{A})\lambda_1(\mathbf{B}).$$

Lidskii's product inequalities and their further developments (see e.g. [2], [3], [8], [11], [12]) generalize (13).

PROOF (cf. the proof of Wang and Zhang [12], Theorem 2). The second part of (12) is an easy consequence of the second part of (15). Hence, assuming $\mathbf{B}$ positive definite,

$$\lambda_{i+l-1}(\mathbf{A}) = \lambda_{i+l-1}(\mathbf{ABB}^{-1}) \le \lambda_{i+l-1-l+1}(\mathbf{AB})\lambda_l(\mathbf{B}^{-1}) = \lambda_i(\mathbf{AB})\lambda_{n-l+1}^{-1}(\mathbf{B}),$$

and the first part of (12) follows. If $\mathbf{B}$ is singular, then continuity argument applies.

COROLLARY 8. Let $\mathbf{A}$ and $\mathbf{B}$ be Hermitian nonnegative definite $n \times n$ matrices. If $1 \le k \le i < j \le n$, $1 \le l \le n - j + 1$, $\lambda_{j+l-1}(\mathbf{A}) > 0$, and $\lambda_{n-l+1}(\mathbf{B}) > 0$, then

$$\operatorname{mls}_{ij} \mathbf{AB} \le \operatorname{mls}_{i-k+1,j+l-1} \mathbf{A} \operatorname{mls}_{k,n-l+1} \mathbf{B}. \tag{14}$$

In particular, if $\mathbf{B}$ is positive definite, then

$$\operatorname{mls}_{ij} \mathbf{AB} \le \operatorname{mls}_{ij} \mathbf{A} \operatorname{mls} \mathbf{B},$$

and if also $\mathbf{A}$ is positive definite, then

$$\operatorname{mls} \mathbf{AB} \le \operatorname{mls} \mathbf{A} \operatorname{mls} \mathbf{B}.$$

PROOF. By (12),

$$\operatorname{mls}_{ij} \mathbf{AB} = \frac{\lambda_i(\mathbf{AB})}{\lambda_j(\mathbf{AB})} \le \frac{\lambda_{i-k+1}(\mathbf{A})\lambda_k(\mathbf{B})}{\lambda_{j+l-1}(\mathbf{A})\lambda_{n-l+1}(\mathbf{B})} = \operatorname{mls}_{i-k+1,j+l-1} \mathbf{A} \operatorname{mls}_{k,n-l+1} \mathbf{B},$$

and (14) is proved.

5 Singular Values of AB

Analogously to Theorem 7, we have

THEOREM 9. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $1 \le k \le i \le n$ and $1 \le l \le n-i+1$, then

$$\sigma_{i+l-1}(\mathbf{A})\sigma_{n-l+1}(\mathbf{B}) \le \sigma_i(\mathbf{AB}) \le \sigma_{i-k+1}(\mathbf{A})\sigma_k(\mathbf{B}). \tag{15}$$

In particular,

$$\sigma_i(\mathbf{A})\sigma_n(\mathbf{B}) \le \sigma_i(\mathbf{AB}) \le \sigma_i(\mathbf{A})\sigma_1(\mathbf{B}) \tag{16}$$

and further

$$\sigma_n(\mathbf{A})\sigma_n(\mathbf{B}) \le \sigma_n(\mathbf{AB}), \qquad \sigma_1(\mathbf{AB}) \le \sigma_1(\mathbf{A})\sigma_1(\mathbf{B}). \tag{17}$$

Gelfand's and Naimark's inequalities and their further developments (see e.g. [2], [3], [8], [10], [11], [12]) generalize (16).

PROOF. For the second part of (15), see e.g. [7], Theorem 3.3.16. For the first part, proceed as in the proof of the first part of (12).

COROLLARY 10. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $1 \le k \le i < j \le n$, $1 \le l \le n - j + 1$, $\sigma_{j+l-1}(\mathbf{A}) > 0$, and $\sigma_{n-l+1}(\mathbf{B}) > 0$, then

$$\operatorname{cnd}_{ij} \mathbf{AB} \le \operatorname{cnd}_{i-k+1,j+l-1} \mathbf{A} \operatorname{cnd}_{k,n-l+1} \mathbf{B}.$$

In particular, if $\mathbf{B}$ is nonsingular, then

$$\operatorname{cnd}_{ij} \mathbf{AB} \le \operatorname{cnd}_{ij} \mathbf{A} \operatorname{cnd} \mathbf{B},$$

and if also $\mathbf{A}$ is nonsingular, then

$$\operatorname{cnd} \mathbf{AB} \le \operatorname{cnd} \mathbf{A} \operatorname{cnd} \mathbf{B}.$$

6 Eigenvalues of $\mathbf{A} \circ \mathbf{B}$

We have the following

THEOREM 11 (see e.g. [5], Theorem 3.1; [7], Theorem 5.3.4). If $\mathbf{A}$ and $\mathbf{B} = (b_{jk})$ are Hermitian nonnegative definite $n \times n$ matrices, then

$$\lambda_n(\mathbf{A})\lambda_n(\mathbf{B}) \le \lambda_n(\mathbf{A}) \min_k b_{kk} \le \lambda_n(\mathbf{A} \circ \mathbf{B})$$

and

$$\lambda_1(\mathbf{A} \circ \mathbf{B}) \le \lambda_1(\mathbf{A}) \max_k b_{kk} \le \lambda_1(\mathbf{A})\lambda_1(\mathbf{B}).$$

COROLLARY 12. If $\mathbf{A}$ and $\mathbf{B}$ are Hermitian positive definite matrices, then

$$\operatorname{mls}(\mathbf{A} \circ \mathbf{B}) \le \operatorname{mls} \mathbf{A} \max_{k,l} \frac{b_{kk}}{b_{ll}} \le \operatorname{mls} \mathbf{A} \operatorname{mls} \mathbf{B}.$$

Sharper inequalities

$$\lambda_i(\mathbf{A})\lambda_n(\mathbf{B}) \le \lambda_i(\mathbf{A} \circ \mathbf{B}) \le \lambda_i(\mathbf{A})\lambda_1(\mathbf{B}),$$

cf. (13), are not generally valid for Hermitian nonnegative definite $n \times n$ matrices. For counterexample, let

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{A} \circ \mathbf{B}.$$

Then $\lambda_1(\mathbf{A})\lambda_3(\mathbf{B}) = 3$ but $\lambda_1(\mathbf{A} \circ \mathbf{B}) = 1$, and $\lambda_2(\mathbf{A} \circ \mathbf{B}) = 1$ but $\lambda_2(\mathbf{A})\lambda_1(\mathbf{B}) = 0$.

But can we sharpen (13) to

$$\lambda_n(\mathbf{A})\lambda_n(\mathbf{B}) \le \lambda_n(\mathbf{A}) \min_k b_{kk} \le \lambda_n(\mathbf{AB}) \tag{18}$$

and

$$\lambda_1(\mathbf{AB}) \le \lambda_1(\mathbf{A}) \max_k b_{kk} \le \lambda_1(\mathbf{A})\lambda_1(\mathbf{B}) \tag{19}$$

for Hermitian nonnegative definite $n \times n$ matrices?

The first inequality of (18) and the second of (19) are elementary facts. To disprove the second inequality of (18), let

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

Then $\lambda_2(\mathbf{AB}) = 0$ but $\lambda_2(\mathbf{A}) \min_k b_{kk} = 1$. To disprove the first inequality of (19), let

$$\mathbf{A} = \begin{pmatrix} 26 & 16 & -11 \\ 16 & 25 & 12 \\ -11 & 12 & 62 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 32 & -13 & -38 \\ -13 & 58 & -2 \\ -38 & -2 & 91 \end{pmatrix}.$$

Then $\lambda_1(\mathbf{AB}) = 7039$ but $\lambda_1(\mathbf{A}) \max_k b_{kk} = 66.64 \cdot 91 = 6064$.

7 Singular Values of $\mathbf{A} \circ \mathbf{B}$

For singular values, we again have some analogy.

THEOREM 13 (see e.g. [5], Theorem 3.1; [7], Theorem 5.5.18). Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. If $\mathbf{P} = (p_{jk}) = (\mathbf{BB}^*)^{\frac{1}{2}}$ and $\mathbf{Q} = (q_{jk}) = (\mathbf{B}^*\mathbf{B})^{\frac{1}{2}}$, then

$$\sigma_1(\mathbf{A} \circ \mathbf{B}) \le \sigma_1(\mathbf{A})\big(\max_k p_{kk} \max_k q_{kk}\big)^{\frac{1}{2}} \le \sigma_1(\mathbf{A})\sigma_1(\mathbf{B}).$$

In particular, if $\mathbf{B} = (b_{jk})$ is Hermitian nonnegative definite, then

$$\sigma_1(\mathbf{A} \circ \mathbf{B}) \le \sigma_1(\mathbf{A}) \max_k b_{kk} \le \sigma_1(\mathbf{A})\sigma_1(\mathbf{B}). \tag{20}$$

The inequality

$$\sigma_n(\mathbf{A})\sigma_n(\mathbf{B}) \le \sigma_n(\mathbf{A} \circ \mathbf{B})$$

is not generally valid. A counterexample is

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It seems that there is no good way to underestimate $\sigma_i(\mathbf{A} \circ \mathbf{B})$ by using singular values of $\mathbf{A}$ and $\mathbf{B}$. Therefore we cannot find bounds for $\mathrm{cnd}_{ij}(\mathbf{A} \circ \mathbf{B})$ by using condition numbers of $\mathbf{A}$ and $\mathbf{B}$.

8 Eigenvalues of AB, Continued

Finally, we consider absolute multiplicative spreads.

It seems that the $|\lambda_{(i)}(\mathbf{AB})|$'s cannot be effectively grasped in general. Since the singular values of a normal matrix are absolute values of eigenvalues, we have by Corollary 10 the following

COROLLARY 14. Let $\mathbf{A}$ and $\mathbf{B}$ be normal $n \times n$ matrices such that also $\mathbf{AB}$ is normal. If $1 \le k \le i < j \le n$, $1 \le l \le n-j+1$, $\lambda_{j+l-1}(\mathbf{A}) \ne 0$, and $\lambda_{n-l+1}(\mathbf{B}) \ne 0$, then

$$\mathrm{Mls}_{ij}\,\mathbf{AB} \le \mathrm{Mls}_{i-k+1,j+l-1}\mathbf{A}\,\mathrm{Mls}_{k,n-l+1}\,\mathbf{B}.$$

In particular, if $\mathbf{B}$ is nonsingular, then

$$\mathrm{Mls}_{ij}\,\mathbf{AB} \le \mathrm{Mls}_{ij}\mathbf{A}\,\mathrm{Mls}\,\mathbf{B},$$

and if also $\mathbf{A}$ is nonsingular, then

$$\mathrm{Mls}\,\mathbf{AB} \le \mathrm{Mls}\,\mathbf{A}\,\mathrm{Mls}\,\mathbf{B}.$$

If $\mathbf{A}$ and $\mathbf{B}$ commute, then analogy to Theorem 7 holds. Namely, applying Yamamoto's theorem

$$\lim_{m\to\infty} \sigma_i(\mathbf{A}^m)^{\frac{1}{m}} = |\lambda_{(i)}(\mathbf{A})| \quad (1 \le i \le n)$$

(see e.g. [7], Theorem 3.3.21) and Theorem 9, we have

THEOREM 15 ([7], Exercise 3.3.30). Let $\mathbf{A}$ and $\mathbf{B}$ be commuting $n \times n$ matrices. If $1 \le k \le i \le n$ and $1 \le l \le n-i+1$, then

$$|\lambda_{(i+l-1)}(\mathbf{A})||\lambda_{(n-l+1)}(\mathbf{B})| \le |\lambda_{(i)}(\mathbf{AB})| \le |\lambda_{(i-k+1)}(\mathbf{A})||\lambda_{(k)}(\mathbf{B})|. \tag{21}$$

In particular,

$$|\lambda_{(i)}(\mathbf{A})||\lambda_{(n)}(\mathbf{B})| \le |\lambda_{(i)}(\mathbf{AB})| \le |\lambda_{(i)}(\mathbf{A})||\lambda_{(1)}(\mathbf{B})|$$

and further

$$|\lambda_{(n)}(\mathbf{A})||\lambda_{(n)}(\mathbf{B})| \le |\lambda_{(n)}(\mathbf{AB})|,\ \ |\lambda_{(1)}(\mathbf{AB})| \le |\lambda_{(1)}(\mathbf{A})||\lambda_{(1)}(\mathbf{B})|.$$

In [7], $\mathbf{A}$ and $\mathbf{B}$ are assumed to be nonsingular in the first part of (21). We prove it in the singular case. Assume that $\mathbf{A}$ or $\mathbf{B}$ is (or both are) singular. Then there exists $\epsilon_0 > 0$ such that $\mathbf{A}_\epsilon = \mathbf{A} + \epsilon\mathbf{I}$ and $\mathbf{B}_\epsilon = \mathbf{B} + \epsilon\mathbf{I}$ are nonsingular for all ϵ with $0 < \epsilon < \epsilon_0$. Because $\mathbf{A}$ and $\mathbf{B}$ commute, also $\mathbf{A}_\epsilon$ and $\mathbf{B}_\epsilon$ commute. Applying the first part of (21) to $\mathbf{A}_\epsilon$ and $\mathbf{B}_\epsilon$, the claim follows by continuity argument.

COROLLARY 16. Let $\mathbf{A}$ and $\mathbf{B}$ be commuting $n \times n$ matrices. If $1 \le k \le i < j \le n$, $1 \le l \le n-j+1$, $\lambda_{j+l-1}(\mathbf{A}) \ne 0$, and $\lambda_{n-l+1}(\mathbf{B}) \ne 0$, then

$$\mathrm{Mls}_{ij}\,\mathbf{AB} \le \mathrm{Mls}_{i-k+1,j+l-1}\mathbf{A}\,\mathrm{Mls}_{k,n-l+1}\,\mathbf{B}.$$

In particular, if **B** is nonsingular, then

$$\mathrm{Mls}_{ij}\,\mathbf{AB} \le \mathrm{Mls}_{ij}\,\mathbf{A}\,\mathrm{Mls}\,\mathbf{B},$$

and if also **A** is nonsingular, then

$$\mathrm{Mls}\,\mathbf{AB} \le \mathrm{Mls}\,\mathbf{A}\,\mathrm{Mls}\,\mathbf{B}.$$

Let **A** and **B** be normal matrices. Then **AB** is not necessarily normal, but it is normal if **A** and **B** commute. However, **AB** can be normal even if **A** and **B** do not commute ([6], Problem 2.5.9). All this motivates us to pose the following

CONJECTURE. Theorem 15 remains valid if, instead of commutativity, normality of **A** and **B** (but not necessarily **AB**) is assumed.

References

[1] G. Alpargu and G. P. H. Styan, Some comments and a bibliography on the Frucht-Kantorovich and Wielandt inequalities. In "Innovations in Multivariate Statistical Analysis: A Festschrift for Heinz Neudecker", Kluwer, 2000, pp. 1–38.

[2] R. Bhatia, Matrix Analysis, Springer, 1997.

[3] R. Bhatia, Linear Algebra to quantum cohomology: The story of Alfred Horn's inequalities, Amer. Math. Monthly 108(2001), 289–318.

[4] G. H. Golub and C. Van Loan, Matrix Computations, 2nd ed., Jonhs Hopkins Univ. Pr., 1989.

[5] R. A. Horn, The Hadamard Product. In "Proc. Sympos. Applied Math., Vol. 40: Matrix Theory and Applications", Amer. Math. Soc., 1990, pp. 87–169.

[6] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge Univ. Pr., 1985.

[7] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge Univ. Pr., 1991.

[8] A. W. Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Applications, Acad. Pr., 1979.

[9] J. K. Merikoski and R. Kumar, Characterizations and lower bounds for the spread of a normal matrix, Linear Algebra Appl. 364(2003), 13–31.

[10] R. C. Thompson, On the singular values of matrix products, Scripta Math. 29(1973), 111–114.

[11] B. I. Wang and B. Y. Xi, Some inequalities for singular values of matrix products, Linear Algebra Appl. 264(1997), 109–115.

[12] B. I. Wang and F. Z. Zhang, Some inequalities for the eigenvalues of the product of positive semidefinite Hermitian matrices. Linear Algebra Appl. 160(1992), 113–118.