Applied Mathematics E-Notes, 25(2025), 1-14 ©
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/ ISSN 1607-2510

Inverse Eigenvalue Problem For Centrosymmetric Matrix: An Optimal Approximate Solution*

Manpreet Kaur† , Vineet Bhatt‡

Received 23 June 2023

Abstract

The left and right inverse eigenvalue problem (IEVP) is a special class of inverse eigenvalue problems (IEVP) that has several applications in engineering and science. However, few authors have studied the left and right IEVP with submatrix constraints. We provide necessary and sufficient conditions along with the general expression to the left and right IEVP with submatrix constraints for centrosymmetric solutions. We also provide the solution of optimal approximation problem for left and right IEVP. For a given arbitrary $(n \times n)$ real matrix $\hat{A}$, we find a unique solution matrix A^* to left and right IEVP such that least Frobenius norm $\|\hat{A} - A^*\|$ is to be obtained, where A^* is centrosymmetric in nature. In addition, we provide an algorithm for calculating the general solution with a numerical example.

1 Introduction

Since the early 19th century, the study of matrix algebra has been a more interesting research topic for researchers in the field of linear algebra. Many researchers have studied a centrosymmetric matrix which is a special type of symmetric matrix [1–4]. A $(n \times n)$ centrosymmetric matrix $A = S_nAS_n$, where S_n is a counter-identity matrix, whose elements are all equal to zero except those on the counter-diagonal, which are all equal to one [5]. Centrosymmetric matrices have several applications in different fields, i.e., communication theory, statistics, physics, harmonic, differential quadrature, differential equation, numerical analysis, engineering, magic square, pattern recognition, Markov process, etc. [6–12]. The symmetric Toeplitz matrix is a special type of the centrosymmetric matrix, in which each descending diagonal from left to right is constant, and appears in digital signal processing and other areas [38, 39]. Eigenvalues and eigenvectors of the centrosymmetric matrix have been helpful within various fields [13, 14]. If T is a linear transformation from a vector space $V(F)$ into itself and $v(\neq 0) \in V$, then v is an eigenvector of T if $T(v) = \lambda(v)$, where λ is scalar in F, known as the eigenvalue [15]. Many researchers have studied inverse eigenvalue problems in the field of linear algebra. An inverse eigenvalue problem deals with the rebuilding of the matrix from fixed data. The spectral data may be composed of the complete or partial information of eigenvalues or eigenvectors. The purpose of the inverse eigenvalue problem is to build a matrix that preserves both a definite special structure and given spectral property [16, 17]. An inverse eigenvalue problem arises in different fields of applications, such as central design, system identification, seismic tomography, principal component analysis, exploration and remote sensing, antenna array processing, geophysics, molecular spectroscopy, physics, structure analysis, circuit theory and mechanic system simulator, etc. [18–24]. Furthermore, the inverse eigenvalue problem (IEVP) plays an important role in the field of linear algebra. It helps in finding the solutions for various matrices like orthogonal matrix, Jacobi matrix, and centrosymmetric matrix [25–29]. From the above analysis, it has been observed that many researchers have studied IEVP for the centrosymmetric matrix under submatrix constraints [25–27]. But few researchers have studied the left and right IEVP for the centrosymmetric matrix [30–32]. The left and right IEVP are a particular class of IEVP, which mostly come in perturbation analysis of matrix eigenvalue, in recursive matrices, and appear in several applications

*Mathematics Subject Classifications: 15A29; 15A09; 15A18; 15A24.

†Department of Mathematics, Chandigarh University, Mohali, Punjab, India
‡Department of Mathematics, Graphic Era Hill University, Uttarakhand, India, corresponding author.

[15, 30–36]. In [30, 33, 34, 15] authors use special properties of eigenpairs of a matrix to solve the left and right IEVP for Skew-symmetric matrices, generalized centrosymmetric matrices, κ Per-symmetric matrices, symmetrizable matrices, orthogonal matrices and κ-Hermitian matrices. In [31, 35] authors have studied the left and right IEVP for real matrices, semi-positive definite matrices, generalized reflexive, anti-reflexive, and (R, S) symmetric matrices with a specific structure of the matrix. Therefore, we study the left and right IEVP for the centrosymmetric matrix under the submatrix principal constraint in this paper. We divide this paper into four sections. The first section contains the introductory part, the second section includes notation and preliminaries, and definitions. The third section includes necessary and sufficient conditions and a general solution matrix to Problem 1, which is discussed in Section 2. In Section 4, we provide the uniqueness theorem of Problem 2, which is discussed in Section 2, and then obtain the unique approximation solution matrix with the orthogonal invariance of the Frobenius norm. In addition, we give an algorithm to compute the unique approximation solution. We conclude the result of the problems in the end.

2 Notations and Preliminaries

In this paper, we use the following notations. Let $R^{m\times n}$ be set of all $m \times n$ real matrices, $C^{m\times n}$ be set of all complex matrices, $R^{m\times n}$ represent the set of all real numbers, $O^{n\times n}$ denote the set of all orthogonal matrices, $CSR^{n\times n}$ denote the set of all $n\times n$ centrosymmetric matrices, $(a_{i,j})$ $(1 \le i \le m, 1 \le j \le m)$, $R(A)$, A^+, A^T, $\rho(A)$ and $tr(A)$ denotes the elements, column space, Moore-Penrose generalized inverse, transpose, rank, and trace of matrix A, respectively. Let 0_n, I_n, S_n be zero matrices of size n, identity matrix of order n, and counter-identity matrix (reverse identity matrix) respectively. For $A, B \in R^{(m\times n)}$, $\langle A, B\rangle = tr(B^TA)$ denotes the inner product of matrices A and B. The Frobenius norm is $\|A\| = \sqrt{\langle A, A\rangle} = \sqrt{tr(A^TA)}$. $R^{m\times n}$ endowed with $\langle \cdot, \cdot\rangle$ is a Hilbert inner product space.

2.1 Basic Definitions

In this section, we provide important definitions related to this paper with appropriate examples. In Definitions 1–3, we construct a centrosymmetric matrix, central principal submatrix and trailing principal submatrix respectively. Furthermore, in Definitions 4–5, we define an orthogonal matrix, left and right eigenpairs, symmetric and anti-symmetric vectors respectively.

Definition 1 *A $(n\times n)$ real matrix A is known as a centrosymmetric matrix if $(a_{i,j}) = (a_{n+1-i,n+1-j}), (1 \le i, j \le n)$.*

For instance,

$$A = \begin{pmatrix} a & b & c \\ d & e & d \\ c & b & a \end{pmatrix}$$

is a (3×3) centrosymmetric matrix.

Definition 2 *A m-square central principal matrix $A_C(m)$ of matrix A is defined as*

$$A_C(m) = (0_{mk}, I_m, 0_{mk})A \begin{pmatrix} 0_{mk} \\ I_m \\ 0_{mk} \end{pmatrix},$$

where 0 is a $(m \times k)$ zero matrix and I is an $(m \times m)$ identity matrix.

For instance, if A is of order 5, then A has no (2×2) central principal submatrices. But A does have (3×3) central principal submatrices situated in the centre of the given matrix, i.e.,

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & \mathbf{b_2} & \mathbf{c_2} & \mathbf{d_2} & e_2 \\ a_3 & \mathbf{b_3} & \mathbf{c_3} & \mathbf{b_3} & a_3 \\ e_2 & \mathbf{d_2} & \mathbf{c_2} & \mathbf{b_2} & a_2 \\ e_1 & d_1 & c_1 & b_1 & a_1 \end{pmatrix}.$$

From above example, it is clearly that the central pincipal matrix $A_C(m)$ of matrix A is also centrosymmetric matrix.

Definition 3 *A m-square trailing principal submatrix $A_t(m)$ is defined as follows:*

$$A_t(m) = (0_m, I_{n-m}, 0_m)A \begin{pmatrix} 0_{m,n-m} \\ I_m \end{pmatrix},$$

where 0 is a $(m \times (n-m))$ zero matrix and I is an $(m \times m)$ identity matrix.

For instance, if A is of order 5, then A has no (2×2) trailing prinicpal submatrices. But A does have (3×3) trailing principal submatrices situated in the left corner of a given matrix as follows:

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & \mathbf{c_3} & \mathbf{b_3} & \mathbf{a_3} \\ e_2 & d_2 & \mathbf{c_2} & \mathbf{b_2} & \mathbf{a_2} \\ e_1 & d_1 & \mathbf{c_1} & \mathbf{b_1} & \mathbf{a_1} \end{pmatrix}.$$

From above example, it is clearly that the trailing principal matrix $A_t(m)$ of matrix A may or may not be a centrosymmetric matrix.

Definition 4 *Matrix $O^{(n \times n)}$ is said to be an orthogonal matrix, if $O^T O = OO^T = I$, I is an $(n \times n)$ identity matrix.*

Definition 5 *Let $x \in R^n$. A vector x is said to be symmetric vector if $S_n x = x$. A vector x is said to be an anti-symmetric vector if $S_n x = -x$.*

Property 1 *For partial left and right eigenpairs (eigenvalues and their corresponding eigenvectors) (λ_i, x_i), $i = 1, 2, ..., h_1$, (μ_j, y_j), $j = 1, 2, ..., h_2$, and a particular $(n \times n)$ matrix set S, matrix $A \in S$ will be derived from equation given below*

$$\begin{array}{ll} Ax_i = \lambda_i x_i & i = 1, 2, ..., h_1, \\ y_j^T A = \mu_j y_j^T & j = 1, 2, ..., h_2, \end{array} \tag{1}$$

where $h_1 \leq m$, $h_2 \leq l$, λ_i, μ_j are eigenvalues, x_i, y_j are corresponding eigenvectors and S is a subspace of $R^{n \times n}$.

If $X = (x_1, x_2, ...x_{h_1}) \in R^{n \times m}$, $\lambda = diag(\lambda_1, \lambda_2, ..., \lambda_{h_1}) \in R^{m \times m}$, $Y = (y_1, y_2, ..., y_{h_2}) \in R^{n \times l}$, $\Gamma = diag(\mu_1, \mu_2, ..., \mu_{h_2}) \in R^{l \times l}$, then (1) is equivalent to

$$AX = X\lambda \text{ and } Y^T A = \Gamma Y^T. \tag{2}$$

Assume that (λ_i, x_i), $i = 1, 2, ..., h_1$ denotes right eigenpairs of A; $(\mu_j, y_j), j = 1, 2, .., h_2$ denotes left eigenpairs of matrix A. The problems studied in this paper may be described as follows:

Problem 1. For X, λ, Y, Γ and $A_0 \in CSR^{k \times k}$, $h_1 \leq m \leq n$, $h_2 \leq l \leq n$, $k \leq n$, find $A \in CSR^{n \times n}$ such that

$$AX = X\lambda, \; Y^T A = \Gamma Y^T \text{ and } A_C(k) = A_0,$$

where $A_C(k)$ be the $(k \times k)$ leading principal submatrix.

Problem 2. Given an arbitrary matrix $\hat{A} \in R^{n\times n}$, find $A^* \in S_A$ such that

$$\left\| A^* - \hat{A} \right\| = \min_{\forall A \in S_A} \left\| A - \hat{A} \right\|,$$

where S_A is the solution set of Problem 1.

3 General Solutions to Problem 1

In this section, we study the central submatrices of the centrosymmetric matrix, which has the same properties and structure as the given centrosymmetric matrix. Therefore, both matrices have similar expressions. In addition, we give the properties of the eigenpairs of centrosymmetric matrices and we have expressed the special form of the eigenvectors of centrosymmetric matrices. Furthermore, we give the necessary and sufficient conditions for the existence of a general solution matrix to Problem 1, which is discussed in Section 2.

Now, e_i is i^{th} ($i \in$ natural numbers) column of I_n, and let $S_n = (e_n, e_{n-1}, ..., e_2, e_1)$. Then $S_n = S_n^T$, $S_n S_n^T = I_n$. Let $k = [\frac{n}{2}]$, where $[\frac{n}{2}]$ is the greatest integer and less than or equal to $\frac{n}{2}$, and let orthogonal matrices be given below:

$$D_n = \begin{cases} \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ S_k & -S_k \end{pmatrix} & \text{if } n = 2k, \\ \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{pmatrix} & \text{if } n = 2k+1. \end{cases}$$

Lemma 1 ([15]) *A matrix A is centrosymmetric of order n iff $S_n A S_n = A$.*

Lemma 2 ([27]) *A matrix $A \in CSR^{n\times n}$, if and only if there exists A_1 and A_2, which are $(n-k)\times(n-k)$ and $(k \times k)$ real matrices, respectively, such that*

$$A = D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T. \tag{3}$$

Lemma 3 *Let $A \in CSR^{n\times n}$ be formed as in equation (3). Then $(k \times k)$ central principal submatrix $A_C(k)$ of A is given below*

$$A_C(k) = D_k \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_k^T, \tag{4}$$

where order of A_1 and A_2 are $((k-t)\times(k-t))$ and $(k \times k)$ respectively.

Proof. If $n = 2r$, from equation (3) and conditions discussed in Definition 2, i.e., a $(2r \times 2r)$ matrix having only central principal submatrices of even order, so

$$A_C(k) = \begin{pmatrix} M & NS_t \\ S_t N & S_t M S_t \end{pmatrix}, \text{where } M, N \in R^{t\times t} \text{ and } k = 2t.$$

Thus,

$$D_k^T A_C(k) D_k = \frac{1}{2} \begin{pmatrix} I_t & S_t \\ I_t & -S_t \end{pmatrix} \begin{pmatrix} M & NS_t \\ S_t & S_t M S_t \end{pmatrix} \begin{pmatrix} I_t & S_t \\ I_t & -S_t \end{pmatrix} = \begin{pmatrix} M+N & 0 \\ 0 & M-N \end{pmatrix}.$$

By setting, $M + N = A_1$, $M - N = A_2$, we obtain the $(k \times k)$ central principal submatrix of $A_C(k)$ as given below

$$A_C(k) = D_k \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_k^T. \tag{5}$$

If $n = 2r + 1$, and a $\{(2r + 1) \times (2r + 1)\}$ matrix has central principal submatrices of odd order, so

$$A_C(k) = \begin{pmatrix} M & u_t & NS_t \\ v_t^T & \alpha & v_t^T S_t \\ S_t N & S_t u_t & S_t N S_t \end{pmatrix},$$

where $M, N \in R^{t\times t}$, $u_t = (0, I_t)u$, $v_t^T = (0, I_t)v$, $k = 2t + 1$. Hence,

$$\begin{aligned} D_k^T A_C(k) D_k &= \frac{1}{2}\begin{pmatrix} I_t & 0 & S_t \\ 0 & \sqrt{2} & 0 \\ I_t & 0 & -S_t \end{pmatrix}\begin{pmatrix} M & u_t & NS_t \\ v_t^T & \alpha & v_t^T S_t \\ S_t N & S_t u_t & S_t N S_t \end{pmatrix}\begin{pmatrix} I_t & 0 & I_t \\ 0 & \sqrt{2} & 0 \\ S_t & 0 & -S_t \end{pmatrix} \\ &= \begin{pmatrix} M+N & \sqrt{2}u_t & 0 \\ \sqrt{2}v_t^T & \alpha & 0 \\ 0 & 0 & M-N \end{pmatrix}. \end{aligned}$$

By setting,

$$\begin{pmatrix} M+N & \sqrt{2}u_t \\ \sqrt{2}v_t^T & \alpha \end{pmatrix} = A_1, \quad M - N = A_2,$$

then $A_C(k)$ may be written as

$$A_C(k) = D_k \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_k^T. \tag{6}$$

By combining equation (5) and equation (6) we get $(k \times k)$ central principal submatrix of A which is given as in equation (4). ■

Lemma 4 *Let $A \in CSR^{n\times n}$ be formed as in equation (3). Then $(k \times k)$ central principal submatrix of A is given below*

$$A_0(k) = D_k \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_k^T, \tag{7}$$

where $A_{10} \in R^{(k-t)\times(k-t)}$ and $A_{20} \in R^{t\times t}$. The matrix $A_0(k)$ is central principal submatrix of order $(k \times k)$ if and if only A_{10} and A_{20} both are trailing principal submatrix of A_1 and A_2, respectively.

Lemma 5 *Assume that $A \in CSR^{n\times n}$ and $(\lambda_i, x_i), (\mu_j, y_j)$ (where $1 \le i \le h_1$, $1 \le j \le h_2$) are right and left real eigenpairs of A, then so are $S_n x_i$, $S_n y_j^T$, $x_i \pm S_n x_i$, $y_j^T \pm S_n y_j^T$.*

Proof. Given $P \in CSR^{n\times n}$, if $(\lambda_i, x_i), (\mu_j, y_j)$ (where $1 \le i \le h_1, 1 \le j \le h_2$) are right and left real eigenpairs respectively, then we get, from Lemma 1,

$$PS_n x_i = S_n P x_i = \lambda_i S_n x_i \;\text{ and }\; y_j^T S_n P = y_j^T P S_n = \mu_j S_n y_j^T.$$

Therefore, $x_i \pm S_n x_i$ are eigenvectors associated with λ_i, where $x_i + S_n x_i$ are symmetric vectors, while $x_i - S_n x_i$ are anti-symmetric vectors. Similarly, $y_j^T + S_n y_j^T$ are symmetric vectors, and $y_j^T - S_n y_j^T$ are anti-symmetric vectors.

If (λ_i, x_i), (μ_j, y_j) (where $1 \le i \le h_1$, $1 \le j \le h_2$) are right and left complex eigenpairs respectively, then we get, from Lemma 1,

$$PS_n \hat{X}_i = S_n P \hat{X}_i = \hat{\lambda}_i S_n \hat{X}_i \;\text{ and }\; \hat{Y}_j^T S_n P = \hat{Y}_j^T P S_n = \hat{\Gamma}_j S_n \hat{Y}_j^T.$$

Thus, $P(\hat{X}_i \pm S_n\hat{X}_i) = (\hat{X}_i \pm S_n\hat{X}_i)\hat{\lambda}_i$ and $(\hat{Y}_j^T \pm \hat{Y}_j^T S_n)P = \hat{\Gamma}_j(\hat{Y}_j^T \pm \hat{Y}_j^T S_n)$, where the columns of $\hat{X}_i + S_n\hat{X}_i = (\xi_i + S_n\xi_i, \eta_i + S_n\eta_i)$ are symmetric vectors, and $\hat{X}_i - S_n\hat{X}_i = (\xi_i - S_n\xi_i, \eta_i - S_n\eta_i)$. Similarly, the columns of $\hat{Y}_j^T + \hat{Y}_j^T S_n = (\xi_i + S_n\xi_i, \eta_i + S_n\eta_i)$ are symmetric vectors, and $\hat{Y}_j^T - \hat{Y}_j^T S_n = (\xi_i - S_n\xi_i, \eta_i - S_n\eta_i)$.

From above analysis, we assume that X, Y and Γ, λ in Problem 1 may be written as given below:

$$X = \begin{pmatrix} \tilde{M}_1 & N_1 \\ S_r\tilde{M}_1 & -S_rN_1 \end{pmatrix}, \quad Y = \begin{pmatrix} \tilde{M}_2 & N_2 \\ S_r\tilde{M}_2 & -S_rN_2 \end{pmatrix}, \quad n = 2r, \tag{8}$$

$$X = \begin{pmatrix} \tilde{M}_1 & N_1 \\ \sqrt{2}c^T & 0 \\ S_r\tilde{M}_1 & -S_rN_1 \end{pmatrix}, \quad Y = \begin{pmatrix} \tilde{M}_2 & N_2 \\ \sqrt{2}d^T & 0 \\ S_r\tilde{M}_2 & -S_rN_2 \end{pmatrix}, \quad n = 2r+1, \tag{9}$$

$$\lambda = diag(\lambda_1, \lambda_2), \quad \Gamma = diag(\Gamma_1, \Gamma_2). \tag{10}$$

where $\tilde{M}_1 \in R^{r\times s_1}$, $N_1 \in R^{r\times(m-s_1)}$, $c \in R^{s_1}$, $\tilde{M}_2 \in R^{r\times s_2}$, $N_2 \in R^{r\times(l-s_2)}$, $d \in R^{s_2}$, $\lambda_1 \in R^{s_1\times s_1}$, $\lambda_2 \in R^{(m-s_1)\times(m-s_1)}$, $\Gamma_1 \in R^{s_2\times s_2}$, $\Gamma_2 \in R^{(l-s_2)\times(l-s_2)}$, where λ_1, λ_2, Γ_1, Γ_2 are block diagonals. Thus, D_n^TX and D_n^TY are given as follows:
If $n = 2r$, then

$$D_n^TX = \frac{1}{\sqrt{2}}\begin{pmatrix} I_r & S_r \\ I_r & -S_r \end{pmatrix}\begin{pmatrix} \tilde{M}_1 & N_1 \\ S_r\tilde{M}_1 & -S_rN_1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\tilde{M}_1 & 0 \\ 0 & \sqrt{2}N_1 \end{pmatrix},$$

and

$$D_n^TY = \frac{1}{\sqrt{2}}\begin{pmatrix} I_r & S_r \\ I_r & -S_r \end{pmatrix}\begin{pmatrix} \tilde{M}_2 & N_2 \\ S_r\tilde{M}_2 & -S_rN_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\tilde{M}_2 & 0 \\ 0 & \sqrt{2}N_2 \end{pmatrix}.$$

If $n = 2r+1$, then

$$D_n^TX = \frac{1}{\sqrt{2}}\begin{pmatrix} I_r & 0 & S_r \\ 0 & \sqrt{2} & 0 \\ I_r & 0 & -S_r \end{pmatrix}\begin{pmatrix} \tilde{M}_1 & N_1 \\ \sqrt{2}c^T & 0 \\ S_r\tilde{M}_1 & -S_rN_1 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\tilde{M}_1 & 0 \\ \sqrt{2}c^T & 0 \\ 0 & \sqrt{2}N_1 \end{pmatrix},$$

and

$$D_n^TY = \frac{1}{\sqrt{2}}\begin{pmatrix} I_r & 0 & S_r \\ 0 & \sqrt{2} & 0 \\ I_r & 0 & -S_r \end{pmatrix}\begin{pmatrix} \tilde{M}_2 & N_2 \\ \sqrt{2}d^T & 0 \\ S_r\tilde{M}_2 & -S_rN_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2}\tilde{M}_2 & 0 \\ \sqrt{2}d^T & 0 \\ 0 & \sqrt{2}N_2 \end{pmatrix}.$$

Now, for $n = 2r$, set $\tilde{M}_1 = M_1$, $\tilde{M}_2 = M_2$, and for $n = 2r+1$, set $M_1 = \begin{pmatrix} \tilde{M}_1 \\ c^T \end{pmatrix}$, $M_2 = \begin{pmatrix} \tilde{M}_2 \\ d^T \end{pmatrix}$, then for all arbitrary n, D_n^TX and D_n^TY may be written in the following form:

$$D_n^TX = \begin{pmatrix} \sqrt{2}M_1 & 0 \\ 0 & \sqrt{2}N_1 \end{pmatrix}, \quad (D_n^TY)^T = \begin{pmatrix} \sqrt{2}M_2 & 0 \\ 0 & \sqrt{2}N_2 \end{pmatrix}^T \tag{11}$$

where $M_1 \in R^{(n-r)\times s_1}$, $N_1 \in R^{r\times(m-s_1)}$, $M_2 \in R^{(n-r)\times s_2}$, $N_2 \in R^{r\times(l-s_2)}$. ∎

Lemma 6 ([27]) *Given $X \in R^{n\times m}$, $Y \in R^{n\times l}$, $\lambda \in R^{m\times m}$, and $\Gamma \in R^{l\times l}$ as in Section 2, then there exists a matrix $A \in R^{n\times n}$ such that*

$$\begin{cases} AX = X\lambda, \\ Y^TA = \Gamma Y^T, \end{cases}$$

if and only if $Y^TXA = \Gamma Y^TX$, $X\lambda X^+X = X\lambda$ and $\Gamma Y^T = Y^+Y\Gamma Y^T$.

In addition, its general solution may be written as:

$$P = X\lambda X^+ + (Y^T)^+\Gamma Y^T(I - XX^+) + Q_1GQ_2^T,$$

where $G \in R^{(n-r_1)\times(n-r_2)}$, $Q_1 \in R^{n\times(n-r_1)}$, ${Q_1}^TQ_1 = I_{n-r_1}$, $r_1 = \rho(Y)$, range space (Q_1) = Null space (Y^T) [37, Lemma 3.7]. Assume that $X \in R^{m\times m}$, $Y \in R^{n\times l}$, $B \in R^{k\times l}$ be given. Denote

$$U_1 \equiv \{A \in R^{m\times n} \mid f_1(A) = \|XAY - B\| = min\}.$$

where *min* shows the matrix norm minimization. Then, every element of U_1 has following form

$$A = X^+BY^+ + G - X^+XGYY^+, \quad \forall G \in R^{m\times n}. \tag{12}$$

In particular, $f_1(A) = 0$ has matrices solutions in $R^{m\times n}$, if and only if $X^+XBYY^+ = B$, and its general solution may be also expressed in the form of equation (12).

Theorem 1 *Partition $A_0 \in CSR^{k\times k}$ as in equation (7). Let $X \in R^{m\times m}$, and $Y \in R^{n\times l}$ be given as in equations (8)–(9), $\lambda \in R^{m\times m}$, and $\Gamma \in R^{l\times l}$ be given as in equation (10). Partition $D_n^T X$ and $D_n^T Y$ as in equation (11). Denote*

$$\begin{aligned} M_0 = M_1\lambda_1 M_1^+ + (N_1^T)^+\Gamma_1 N_1^T(I_{k-t} - M_1M_1^+), \quad N_0 = M_2\lambda_2 M_2^+ + (N_2^T)^+\Gamma_2 N_2^T(I_t - M_2M_2^+); \\ H_1 = (0, I_{k-t})Q_3, \quad H_2 = Q_4^T(0, I_{k-t})^T, \quad H_3 = (0, I_t)Q_4, \quad H_4 = Q_5^T(0, I_t)^T; \\ K_1 = A_{10} - (0, I_{k-t})M_0(0, I_{k-t})^T, \quad K_2 = A_{20} - (0, I_t)N_0(0, I_t)^T, \end{aligned} \tag{13}$$

where, $Q_3 \in R^{(n-r)\times(n-r-r_3)}$, $r_3 = rank(N_1)$, $Q_4 \in R^{(n-r)\times(n-r-r_4)}$, $r_4 = rank(M_1)$, $Q_5 \in R^{r\times(r-r_5)}$, $r_5 = rank(N_2)$, $Q_6 \in R^{r\times(r-r_6)}$, $r_6 = rank(M_2)$, range space (Q_3) = null space (N_1^T), range space (Q_4) = null space (M_1^T), range space (Q_5) = null space (N_2^T), range space (Q_6) = null space (M_2^T);

$$Q_3^TQ_3 = I_{n-r-r_3}, \quad Q_4^TQ_4 = I_{n-r-r_4}, \quad Q_5^TQ_5 = I_{r-r_5}, \quad Q_6^TQ_6 = I_{r-r_6}. \tag{14}$$

Then, Problem 1 is solvable if and only if

$$N_1^TM_1\lambda_1 = \Gamma_1N_1^TM_1, \quad M_1\lambda_1M_1^+M_1 = M_1\lambda_1, \quad \Gamma_1N_1^T = N_1^+N_1\Gamma_1N_1^T, \tag{15}$$

$$N_2^TM_2\lambda_2 = \Gamma_2N_2^TM_2, \quad M_2\lambda_2M_2^+M_2 = M_2\lambda_2, \quad \Gamma_2N_2^T = N_2^+N_2\Gamma_2N_2^T, \tag{16}$$

$$H_1H_1^+K_1H_2^+H_2 = K_1, \quad H_3H_3^+K_2H_4^+H_4 = K_2. \tag{17}$$

In addition, every matrix $A \in S_A$ may be expressed as

$$A = D_n \begin{pmatrix} M_0 + Q_3G_1Q_4^T & 0 \\ 0 & N_0 + Q_5G_2Q_6^T \end{pmatrix} D_n^T, \tag{18}$$

where

$$\begin{cases} G_1 = H_1^+K_1H_2^+ + G_3 - H_1^+H_1G_3H_2H_2^+, \\ G_2 = H_3^+K_2H_4^+ + G_4 - H_3^+H_3G_4H_4H_4^+, \\ G_3 \in R^{(n-r-r_3)\times(n-r-r_4)}, \\ G_4 \in R^{(r-r_5)\times(r-r_6)}, \end{cases} \tag{19}$$

are arbitrary.

Proof. From Lemmas 2 and 3, Problem 1 is equivalent to evaluating $A_1 \in R^{(n-r)\times(n-r)}$ and $A_2 \in R^{r\times r}$, such that

$$A = D_n \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} D_n^T, \tag{20}$$

where, A_1 and A_2 satisfy

$$\begin{cases} A_1M_1 = M_1\lambda_1, \\ Y_1^TA_1 = \Gamma_1Y_1^T, \end{cases} \quad \begin{cases} A_2M_2 = M_2\lambda_2, \\ Y_2^TA_2 = \Gamma_2Y_2^T, \end{cases}$$

$$A_{10} = A_1[k-t] = (0, I_{k-t})A_1(0, I_{k-t})^T,$$

and

$$A_{20} = A_2[t] = (0, I_t)A_2(0, I_t)^T.$$

By Lemma 6, we know that the equation (20) holds if and only if,

$$N_1^T M_1 \lambda_1 = \Gamma_1 N_1^T M_1, \quad M_1 \lambda_1 M_1^+ M_1 = M_1 \lambda_1, \quad \Gamma_1 N_1^T = N_1^+ N_1 \Gamma_1 N_1^T, \tag{21}$$

$$N_2^T M_2 \lambda_2 = \Gamma_2 N_2^T M_2, \quad M_2 \lambda_2 M_2^+ M_2 = M_2 \lambda_2, \quad \Gamma_2 N_2^T = N_2^+ N_2 \Gamma_2 N_2^T, \tag{22}$$

which means that equation (16) holds. Furthermore, A_1 and A_2 can be expressed as

$$A_1 = M_0 + Q_3 G_1 Q_4^T, \; A_2 = N_0 + Q_5 G_2 Q_6^T, \tag{23}$$

where $G_1 \in R^{(n-r-r_3)\times(n-r-r_4)}$ and $G_2 \in R^{(r-r_5)\times(r-r_6)}$ are arbitrary real matries.

Now, using the definitions of K_1, K_2, H_1, H_2, H_3 and H_4 in equations (13) and substitute (21)–(23) into (23), then

$$H_1 G_1 H_2 = K_1, \quad H_3 G_2 H_4 = K_2. \tag{24}$$

From Lemma 6, equation 24 holds

$$H_1 H_1^+ K_1 H_2^+ H_2 = K_1, \quad H_3 H_3^+ K_2 H_4^+ H_4 = K_2,$$

which implies that equation (17) holds, and G_1, G_2 may be written as

$$G_1 = H_1^+ K_1 H_2^+ + G_3 - H_1^+ H_1 G_3 H_2 H_2^+,$$

$$G_2 = H_3^+ K_2 H_4^+ + G_4 - H_3^+ H_3 G_4 H_4 H_4^+,$$

where, $G_3 \in R^{(n-r-r_3)\times(n-r-r_4)}$ and $G_4 \in R^{(r-r_5)\times(r-r_6)}$ are arbitrary.

Therefore, the solution to Problem 1 has the form of equation (18)

$$A = D_n \begin{pmatrix} M_0 + Q_3 G_1 Q_4^T & 0 \\ 0 & N_0 + Q_5 G_2 Q_6^T \end{pmatrix} D_n^T.$$

■

4 The Optimal Approximation Solution

In this section, we provide the uniqueness theorem of Problem 2 which is discussed in Section 2 and also provide an unique approximation solution with the Frobenius norm. In addition, we provide an algorithm to evaluate the unique solution. From equation (18), it is easily proved that the solution set S_A is a nonempty closed convex set. Thus,the optimal approximation problem has an unique solution as follows: let us consider singular decompositions of H_1, H_2, H_3, H_4 given in equation (13) as

$$H_1 = U_1 \begin{pmatrix} \sum_1 & 0 \\ 0 & 0 \end{pmatrix} V_1^T, \; H_2 = U_2 \begin{pmatrix} \sum_2 & 0 \\ 0 & 0 \end{pmatrix} V_2^T, \; H_3 = U_3 \begin{pmatrix} \sum_3 & 0 \\ 0 & 0 \end{pmatrix} V_3^T \text{ and } H_4 = U_4 \begin{pmatrix} \sum_4 & 0 \\ 0 & 0 \end{pmatrix} V_4^T, \tag{25}$$

where $q_i = \rho(H_1)$, $\sigma_i = diag(\sigma_1^{(i)}, \sigma_2^{(i)}, \ldots, \sigma_{q_i}^{(i)})$, $q_j^{(i)} > 0$, $(j = 1, \ldots, q_i; i = 1, \ldots, 4)$; $U_1 = (U_{11}, U_{12}) \in O^{(k-t)\times(k-t)}$, $V_1 = (V_{11}, V_{12}) \in O^{(n-r-r_3)\times(n-r-r_3)}$, $U_{11} \in R^{(k-t)\times q_1}$, $V_{11} \in R^{(n-r-r_3)\times q_1}$, $U_2 = (U_{21}, U_{22}) \in O^{(n-r-r_4)\times(n-r-r_4)}$, $V_2 = (V_{21}, V_{22}) \in O^{(k-t)\times(k-t)}$, $U_{21} \in R^{(n-r-r_4)\times q_2}$, $V_{21} \in R^{(k-t)\times q_2}$, $U_3 = (U_{31}, U_{32}) \in O^{t\times t}$, $V_3 = (V_{31}, V_{32}) \in O^{(r-r_5)\times(r-r_5)}$, $U_{31} \in R^{t\times q_3}$, $V_{31} \in R^{(r-r_5)\times q_3}$, $U_4 = (U_{41}, U_{42}) \in O^{(r-r_6)\times(r-r_6)}$, $V_4 = (V_{41}, V_{42}) \in O^{t\times t}$, $U_{41} \in R^{(r-r_6)\times q_4}$, $V_{41} \in R^{t\times q_4}$.

Theorem 2 *Given an arbitrary $\hat{A} \in R^{n\times n}$ and assume that the singular decompositions of H_1,H_2,H_3,H_4 have the forms given in equation (25). Partition the matrix $D_n^T \hat{A} D_n$ as follows:*

$$D_n^T \hat{A} D_n = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}, \tag{26}$$

where $\hat{A}_{11} \in R^{(n-r)\times(n-r)}$, $\hat{A}_{22} \in R^{r\times r}$. *If* $S_A \neq \phi$, *then Problem 2 has a unique solution as follows*

$$A^* = D_n \begin{pmatrix} M_0 + Q_3 G_1 Q_4^T & 0 \\ 0 & N_0 + Q_5 G_2 Q_6^T \end{pmatrix} D_n^T, \tag{27}$$

where

$$G_1 = H_1^+ K_1 H_2^+ + Q_3^T \hat{A}_{11} Q_4 - H_1^+ H_1 Q_3^T \hat{A}_{11} Q_4 H_2 H_2^+$$

and

$$G_2 = H_3^+ K_2 H_4^+ + Q_5^T \hat{A}_{22} Q_6 - H_3^+ H_3 Q_5^T \hat{A}_{22} Q_6 H_4 H_4^+.$$

Proof. Let A be an arbitrary solution in S_A. Then from equation (18)

$$\begin{aligned} \|A - \hat{A}\|^2 &= \left\| D_n \begin{pmatrix} M_0 + Q_3 G_1 Q_4^T & 0 \\ 0 & N_0 + Q_5 G_2 Q_6^T \end{pmatrix} D_n^T - \hat{A} \right\|^2 \\ &= \|M_0 + Q_3 G_1 Q_4^T - \hat{A}_{11}\|^2 + \|N_0 + Q_5 G_2 Q_6^T - \hat{A}_{22}\|^2 + \|\hat{A}_{12}\|^2 + \|\hat{A}_{21}\|^2. \end{aligned} \tag{28}$$

Thus, $\|A - \hat{A}\| = min_{A\in S_A}$, if and only if

$$\begin{cases} \|Q_3 G_1 Q_4^T - (\hat{A}_{11} - M_0)\| = \min_{G_1 \in R^{(n-r-s_3)\times(n-r-s_4)}}, \\ \|Q_5 G_2 Q_6^T - (\hat{A}_{22} - N_0)\| = \min_{G_2 \in R^{(r-s_5)\times(r-s_6)}}. \end{cases} \tag{29}$$

From equation (14), we have that $Q_3^T(N_1^T)^+ = 0$, $Q_5^T(N_2^T)^+ = 0$, $M_1^+ Q_4 = 0$ and $M_2^+ Q_6 = 0$. Thus, from equation (13), the definitions of M_0 and N_0, we get $Q_3^+ M_0 Q_4 = 0$ and $Q_5^+ N_0 Q_6 = 0$. Thus, equations (15) and (29) are equivalent to

$$\begin{cases} \|G_1 - Q_3^T \hat{A}_{11} Q_4\| = \min_{G_1 \in R^{(n-r-s_3)\times(n-r-s_4)}}, \\ \|G_2 - Q_5^T \hat{A}_{22} Q_6\| = \min_{G_2 \in R^{(r-s_5)\times(r-s_6)}}. \end{cases} \tag{30}$$

Let us consider the partition

$$V_1^T G_3 U_2 = \begin{pmatrix} G_{31} & G_{32} \\ G_{33} & G_{34} \end{pmatrix} \quad \text{where } G_3 \in R^{q_1\times q_2}. \tag{31}$$

From equation (25), $V_{12}^+ H_1^+ = 0$, $H_2^+ U_{22} = 0$, then we derive by using equation (19),

$$\begin{aligned} \|G_1 - Q_3^T \hat{A}_{11} Q_4\|^2 &= \|G_3 - H_1^+ H_1 G_3 H_2 H_2^+ - (Q_3^T \hat{A}_{11} Q_4 - H_1^+ K_1 H_2^+)\|^2 \\ &= \| \begin{pmatrix} G_{31} & G_{32} \\ G_{33} & G_{34} \end{pmatrix} - \begin{pmatrix} I_{q_1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} G_{31} & G_{32} \\ G_{33} & G_{34} \end{pmatrix} \begin{pmatrix} I_{q_1} & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad - V_1^T (Q_3^T \hat{A}_{11} Q_4 - H_1^+ K_1 H_2^+) U_2\|^2 \\ &= \left\| \begin{pmatrix} -V_{11}^T (Q_3^T \hat{A}_{11} Q_4 - H_1^+ K_1 H_2^+) U_{21} & G_{32} - V_{11}^T Q_3^T \hat{A}_{11} Q_4 U_{22} \\ G_{33} - V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{21} & G_{34} - V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{22} \end{pmatrix} \right\|^2 \\ &= \|V_{11}^T (Q_3^T \hat{A}_{11} Q_4 - H_1^+ K_1 H_2^+) U_{21}\|^2 + \|G_{32} - V_{11}^T Q_3^T \hat{A}_{11} Q_4 U_{22}\|^2 \\ &\quad + \|G_{33} - V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{21}\|^2 + \|G_{34} - V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{22}\|^2. \end{aligned}$$

Hence, equation (29) holds if and only if

$$G_{32} = V_{11}^T Q_3^T \hat{A}_{11} Q_4 U_{22}, \quad G_{33} = V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{21}, \quad G_{34} = V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{22},$$

Hence, equation (31) becomes

$$G_3 = V_1 \begin{pmatrix} G_{31} & V_{11}^T Q_3^T \hat{A}_{11} Q_4 U_{22} \\ V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{21} & V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{22} \end{pmatrix} U_2^T,$$

where $G_{31} \in R^{q_1 \times q_2}$ is an arbitrary matrix.

Similarly, equation (29) holds if and only if

$$G_4 = V_3 \begin{pmatrix} G_{41} & V_{31}^T Q_5^T \hat{A}_{22} Q_6 U_{42} \\ V_{32}^T Q_5^T \hat{A}_{22} Q_6 U_{41} & V_{32}^T Q_5^T \hat{A}_{22} Q_6 U_{42} \end{pmatrix} U_4^T,$$

where $G_{41} \in R^{q_3 \times q_4}$ is an arbitrary matrix.

Putting G_3 and G_4 into equation (19) and using equation (25), we get

$$\begin{aligned} G_1 &= H_1^+ K_1 H_2^+ + V_1 \begin{pmatrix} G_{31} & V_{11}^T Q_3^T \hat{A}_{11} Q_4 U_{22} \\ V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{21} & V_{12}^T Q_3^T \hat{A}_{11} Q_4 U_{22} \end{pmatrix} U_2^T \\ &= H_1^+ K_1 H_2^+ + Q_3^T \hat{A}_{11} Q_4 - H_1^+ H_1 Q_3^T \hat{A}_{11} Q_4 H_2 H_2^+ \end{aligned}$$

and

$$G_2 = H_3^+ K_2 H_4^+ + Q_5^T \hat{A}_{22} Q_6 - H_3^+ H_3 Q_5^T \hat{A}_{22} Q_6 H_4 H_4^+.$$

The solution A^* in equation (27) is an unique solution of Problem 2 which is the element of solution set of Problem 1 i.e., $A^* \in S_A$. For any arbitrary matrix $\hat{A} \in R^{n \times n}$ the value of $\left\| A^* - \hat{A} \right\|$ is equal to minimum of $\left\| A - \hat{A} \right\|$ for all $A \in S_A$. It clearly shows that the unique solution to the optimal approximation problem has the same form as given in equation (27). ■

Now, we provide an algorithm to evaluate A^* of optimal approximation problem and give a numerical example.

Algorithm 1
1. *Input λ, Γ, X, Y, A_0, and $\hat{A}$, where X, Y, and λ, Γ are given in equation (8) and equation (10), respectively. Also get λ_1, λ_2, Γ_1,Γ_2 from equation (10).*
2. *Compute M_1,M_2, N_1, N_2 from equation (11).*
3. *Construct Q_3, Q_4, Q_5, Q_6 have formed as in equation (14) and satisfies equation (15).*
4. *Follow equation (13) to calculate M_0, N_0, H_1, H_2, H_3, H_4, K_1, K_2.*
5. *If equation (16) and equation (17) holds, then compute; otherwise, stop.*
6. *Derive $\hat{A}_{11}$, $\hat{A}_{22}$ according to equation (26).*
7. *Calculate A^* in light of equation (27)*

Example 1 *Assume that $n = 10$, $m = 5$, $l = 4$, $k = 4$ and let*

$$X = \begin{pmatrix} -0.1940 & -0.2040 & -0.1525 & 0.0289 & -0.1827 \\ -0.2201 & 0.3856 & -0.2383 & 0.0178 & -0.5523 \\ -0.2403 & -0.0444 & 0.5239 & 0 & -0.2435 \\ 0.4128 & 0.5009 & 0.2133 & -0.1773 & 0.3090 \\ -0.4310 & 0.2385 & -0.2205 & -0.1370 & 0.0825 \\ -0.4310 & 0.2385 & 0.2205 & 0.1370 & -0.0825 \\ 0.4128 & 0.5009 & -0.2133 & 0.1773 & -0.3090 \\ -0.2403 & -0.0444 & -0.5239 & 0 & 0.2435 \\ -0.2201 & 0.3856 & 0.2383 & -0.0178 & 0.5523 \\ -0.1940 & -0.2040 & 0.1525 & -0.0289 & 0.1827 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0.3003 & 0.1599 & -0.5677 & 0 \\ -0.4007 & 0.0851 & 0.2460 & -0.1018 \\ 0.1443 & 0.5989 & -0.1283 & 0.0967 \\ -0.3396 & 0.1831 & 0.0401 & -0.1561 \\ 0.3362 & -0.2739 & -0.1395 & 0.1888 \\ -0.3362 & -0.2739 & 0.1395 & -0.1888 \\ -0.3396 & 0.1831 & -0.0401 & 0.1561 \\ 0.1443 & 0.5989 & 0.1283 & -0.0967 \\ -0.4007 & 0.0851 & -0.2460 & 0.1018 \\ 0.3003 & 0.15990.1599 & 0.5677 & 0 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} -0.1500 & -0.3800 & 1.7900 & 0.8600 \\ 0.5700 & -0.6950 & -0.1050 & 0.7200 \\ 0.7200 & -0.1050 & -0.6950 & 0.5700 \\ 0.8600 & 1.7900 & -0.3800 & -0.1500 \end{pmatrix},$$

$$\bigwedge = \begin{pmatrix} -0.2996 & 0 & 0 & 0 & 0 \\ 0 & 2.8021 & 0 & 0 & 0 \\ 0 & 0 & -1.8519 & 0.6464 & 0 \\ 0 & 0 & -0.6464 & -1.8519 & 0 \\ 0 & 0 & 0 & 0 & 1.3045 \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} -2.8021 & 0 & 0 & 0 \\ 0 & 1.4094 & 0 & 0 \\ 0 & 0 & 0.7297 & -0.1497 \\ 0 & 0 & 0.14997 & 0.7297 \end{pmatrix},$$

$$\hat{A} = \begin{pmatrix} 0.0657 & 2.0075 & 0.3873 & 0.4359 & 2.8338 & -0.2362 & 2.6122 & 2.6671 & 1.2700 & 1.0236 \\ 0.5056 & 1.9765 & 2.5052 & 1.4702 & 1.6393 & -0.6379 & -0.3746 & 1.6080 & -0.0465 & 0.7979 \\ 0.7621 & -0.5224 & 2.9527 & -0.7588 & -0.1227 & 0.5337 & 2.1803 & 2.5570 & 0.9993 & 0.2150 \\ 2.0318 & 1.5027 & -0.6027 & -0.7284 & 2.3471 & -0.8447 & 1.3917 & 2.7394 & 0.5252 & 2.8008 \\ -0.9709 & 1.8532 & 1.5550 & 1.8680 & 2.3362 & 2.6468 & 0.7396 & 2.3321 & 0.5615 & -0.4634 \\ -0.4634 & 0.5615 & 2.3321 & 0.7396 & 2.6468 & 2.3362 & 1.8680 & 1.5550 & 1.8532 & -0.9709 \\ 2.8008 & 0.5252 & 2.7394 & 1.3917 & -0.8447 & 2.3471 & -0.7284 & -0.6027 & 1.5027 & 2.0318 \\ 0.2150 & 0.9993 & 2.5570 & 2.1003 & 0.5337 & -0.1220 & -0.7588 & 2.9527 & -0.5224 & 0.7621 \\ 0.7979 & -0.0465 & 1.6080 & -0.3746 & -0.6379 & 1.4702 & 1.4702 & 2.5052 & 1.9765 & 0.5056 \\ 1.2036 & 1.2700 & 2.6671 & 2.6122 & -0.2362 & 2.8338 & 0.4359 & 0.3873 & 2.0075 & 0.0657 \end{pmatrix}.$$

The unique optimal approximation centrosymmetric solution of Problem 2 is

$$A^{*} = \begin{pmatrix} 0.1508 & -1.1821 & 0.2580 & 0.8433 & -0.3736 & -0.8136 & 0.0293 & -0.5635 & 3.0517 & 0.8846 \\ -0.4766 & 0.4265 & 0.1178 & 1.1871 & 0.1878 & -0.5211 & 0.3468 & -0.6498 & 2.2080 & 0.4536 \\ -0.4341 & -1.0929 & 1.3792 & -0.4953 & -1.3551 & 0.3625 & 0.3682 & 0.2613 & 1.4391 & 0.5285 \\ 0.1325 & 0.7412 & 0.2174 & -0.1500 & -0.3800 & 1.7900 & 0.8600 & 0.2651 & 4.7715 & 0.9564 \\ -1.7722 & -3.4805 & -0.2438 & 0.5700 & -0.6950 & -0.1050 & 0.7200 & -0.7233 & 6.3473 & 1.8046 \\ 1.8046 & 6.3473 & -0.7233 & 0.7200 & -0.1050 & -0.6950 & 0.5700 & -0.2438 & -3.4805 & -1.7722 \\ 0.9564 & 4.7715 & 0.2651 & 0.8600 & 1.7900 & -0.3800 & -0.1500 & 0.2174 & 0.7412 & 0.1325 \\ 0.5285 & 1.4391 & 0.2613 & 0.3682 & 0.3625 & -1.3551 & -0.4953 & 1.3792 & -1.0929 & -0.4341 \\ 0.4536 & 2.2080 & -0.6498 & 0.3468 & -0.5211 & 0.1878 & 1.1871 & 0.1178 & 0.4265 & -0.4766 \\ 0.8846 & 3.0517 & -0.5635 & 0.0293 & -0.8136 & -0.3736 & 0.8433 & 0.2580 & -1.1821 & 0.1508 \end{pmatrix}.$$

We also have $A^{*}_{C}(4) = A_0$. *Above example proved that Algorithm 1 is feasible. Furthermore, we also have* $\left\| A^{*} - \hat{A} \right\| = 21.1711.$

5 Conclusion

In this paper, we have studied the left and right IEVP for generalized centrosymmetric matrices. We have proved the necessary and sufficient conditions of Problem 1, which is discussed in Section 2. We also expressed the general solution matrix of Problem 1. For any arbitrary $\hat{A} \in R^{n\times n}$, we have obtained a unique solution $A^* \in CSR^{n\times n}$ for Problem 2 which is discussed in Section 2. We have proposed an algorithm for finding the best approximation solution for generalized centrosymmetric matrices. A numerical experiment proves the effectiveness of results of this article.

Acknowledgment. The authors express their hearty thanks and gratefulness to all those scientists whose masterpieces have been consulted during the preparation of the present research article. We are also thankful and grateful to all reviewers and Editor in Chief of Applied Mathematics E-Notes whose valuable comments have improved this article.

References

[1] L. Datta and S. D. Morgera, On the reducibility of centrosymmetric matrices–applications in engineering problems, Circuits Systems Signal Process., 8(1989), 71–96.

[2] J. R. Weaver, Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues, and eigenvectors, Amer. Math. Monthly, 92(1985), 711–717.

[3] A. R. Collar, On centrosymmetric and centroskew matrices, Quart. J. Mech. Appl. Math., 15(1962), 265–281.

[4] A. L. Andrew, Classroom note: Centrosymmetric matrices, SIAM Rev., 40(1998), 697–698.

[5] I. T. Abu-Jeib, On the counteridentity matrix, Missouri Journal of Mathematical Sciences, 17(2005), 26–34.

[6] W. Chen, Y. Yu and X. Wang, Reducing the computational requirements of the differential quadrature method, Numer. Methods Partial Differential Equations, 12(1996), 565–577.

[7] J. R. Doner and V. R. R. Uppuluri, A Markov chain structure for riff shuffling, SIAM J. Appl. Math., 18(1970), 191–209.

[8] G. W. Adamson and J. Boreham, The use of an association measure based on character structure to identify semantically related pairs of words and document titles, Information storage and retrieval, 10(1974), 253–260.

[9] C. R. Paul and A. E. Feather, Application of moment methods to the characterization of ribbon cables, Computers & Electrical Engineering, 4(1977), 173–184.

[10] V. Colombo, S. E. Corno and P. Ravetto, A new application of space asymptotic transport theory, Annals of Nuclear Energy, 15(1988), 533–542.

[11] S. L. Goud, V. Balasubramanian and B. K. Gupta, Green's function application to chemical perturbation studies, Proceedings of the Indian Academy of Sciences-Section A, Chemical Sciences, 87(1978), 353–358.

[12] A. Cantoni and P. Butler, Properties of the eigenvectors of persymmetric matrices with applications to communication theory, IEEE Transactions on Communications, 24(1976), 804–809.

[13] A. Cantoni and P. Butler, Eigenvalues and eigenvectors of symmetric centrosymmetric matrices, Linear Algebra Appl., 13(1976), 275–288.

[14] N. Muthiyalu and S. Usha, Eigenvalues of centrosymmetric matrices, Computing, 48(1992), 213–218.

[15] G. H. Golub and C. F. Van loan, Matrix Computations, Johns Hopkins University Press, Baltimore, MD, 2013.

[16] V. Barcilon, Inverse eigenvalue problems. In: Talenti, G. (eds) Inverse Problems. lecture Notes in Mathematics, vol. 1225. Springer, Berlin, Heidelberg, 1986.

[17] S. Friedland, Inverse eigenvalue problems, Linear Algebra Appl., 17(1977), 15–51.

[18] M. T. Chu, Inverse eigenvalue problems, SIAM Rev., 40(1998), 1–39.

[19] K. T. Joseph, Inverse eigenvalue problem in structural design, AIAA journal, 30(1992), 2890–2896.

[20] S. Friedland, J. Nocedal and M. L. Overton, The formulation and analysis of numerical methods for inverse eigenvalue problems, SIAM J. Numer. Anal., 24(1987), 634–667.

[21] G. M. L. Gladwell, Inverse Problems in Vibration, Springer Science and Business Media, 2006.

[22] C. B. Smith, and E. M. Hernandez, Non-negative constrained inverse eigenvalue problems–Application to damage identification, Mechanical Systems and Signal Processing, 129(2019), 629–644.

[23] T. Weymuth and M. Reiher, Inverse quantum chemistry: Concepts and strategies for rational compound design, International Journal of Quantum Chemistry, 114(2014), 823–837.

[24] Y. X. Yuan and H. Dai, A generalized inverse eigenvalue problem in structural dynamic model updating, J. Comput. Appl. Math., 226(2009), 42–49.

[25] Z. J. Bai, The inverse eigenproblem of centrosymmetric matrices with a submatrix constraint and its approximation, SIAM J. Matrix Anal. Appl., 26(2005), 1100–1114.

[26] F. Z. Zhou, X. Y. Hu and L. Zhang, The solvability conditions for the inverse eigenvalue problems of centro-symmetric matrices, Linear Algebra Appl., 364(2003), 147–160.

[27] Z. Y. Peng, X. Y. Hu and L. Zhang, The inverse problem of centrosymmetric matrices with a submatrix constraint, J. Comput. Math., 22(2004), 535–544.

[28] H. Šmigoc, The inverse eigenvalue problem for nonnegative matrices, Linear Algebra Appl., 393(2004), 365–374.

[29] O. H. Hald, Inverse eigenvalue problems for Jacobi matrices, Linear Algebra Appl., 14(1976), 63–85.

[30] N. J. Vickers, Animal communication: When i'm calling you, Will you answer too?, Current biology, 27(2017), R713–R715.

[31] K. G. Woodgate, Least-squares solution of $F = PG$ over positive semidefinite symmetric P, linear algebra and its applications, 245(1996), 171–190.

[32] F. L. Li, Left and right inverse eigenpairs problem with a submatrix constraint for the generalized centrosymmetric matrix, Open Math., 18(2020), 603–615.

[33] F. L. Li, X. Y. Hu and L. Zhang, Left and right inverse eigenpairs problem for κ-Hermitian matrices, J. Appl. Math., (2013), 6 pp.

[34] F. Li, Left and right inverse eigenpairs problem of orthogonal matrices, Applied Mathematics, (2012), 1972–1976.

[35] M. l. Liang and L. F. Dai, The left and right inverse eigenvalue problems of generalized reflexive and anti-reflexive matrices, J. Comput. Appl. Math., 234(2010), 743–749.

[36] F. L. Li, X. Y. Hu and L. Zhang, Left and right inverse eigenpairs problem of skew-centrosymmetric matrices, Appl. Math. Comput., 177(2006), 105–110.

[37] X. Y. Hu and L. Zhang, Least-square approximate solutions of a class of matrix problems, J. Hunan Univ, 17(1990), 98–102.

[38] P. Amodio and L. Brugnano, The conditioning of Toeplitz band matrices, Math. Comput. Modelling, 23(1996), 29–42.

[39] V. Lakshmikantham and D. Trigiante, Theory of Difference Equations Numerical Methods and Applications, second ed., CRC Press, 2002.