Conformal Invariants: Perspectives from Geometric PDE

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Gaussian Curvature on compact surface

- Denote K_g the Gaussian curvature on a compact surface (M^2, g),
  Gauss Bonnet formula

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- Uniformization Theorem for compact surface: To classify compact surfaces according to the sign of $\int K_g dv_g$.
  Main tool: Denote $g_w = e^{2w}g$, a conformal change of the metric g, solve

$$-\Delta_g w + K_g = K_{g_w} e^{2w} \ on \ M$$

  for $K_{g_w} =$ constant, where the sign of the constant is the same as $\int K_g dv_g$

Integral conformal invariant

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- The main goal is to find such integral conformal invariants, and the role they play in the uniformization theorem on manifolds of higher dimension..

Outline of talk

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- §4. Renormalized volume on CFT/ADS Conformal compact manifolds.
  Viaclovsky '00, Chang-Gursky-Yang '02 Gursky-Viaclovsky, '04
  Chang-Qing-Yang '06
  Work of
  Chang-Fang '08, R. Graham '08, Juhl, '11
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- §5. Compactness results on conformal compact manifolds
  Chang-Yang '11, Chang-Ge-Yang (WIP)

Gauss-Bonnet integrand on 4-manifold

- Gauss-Bonnet-Chern Formula:

$$8\pi^2\chi(M^4) = \int (\frac{1}{4}|W_g|^2 + \frac{1}{6}({R_g}^2 - 3|Ric|_g^2))dv_g \tag{1}$$

where W denotes the Weyl tensor.

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  where W denotes the Weyl tensor.
- $W_{g_w} = e^{-2w}W_g$ implies $|W_g|^2 dv_g = |W_{g_w}|^2 dv_{g_w}$ Thus the integral $\int \frac{1}{6}({R_g}^2 - 3|Ric|_g^2))dv_g$ is a conformal invariant.

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- Viaclovsky '00 studied symmetric function of Schouten tensor:

$$A = \frac{1}{n-2}(Ric - \frac{R}{2(n-1)}g). \qquad (2)$$

which is natural to study as (M^n, g)

$$Rm = W_g \oplus A \owedge g, \qquad (3)$$

and W_g is a pointwise conformal invariant.

Elementary symmetric function

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- Examples:

$$\begin{aligned} \sigma_1(A_g) &= \sum_i \lambda_i = \frac{n-2}{2(n-1)} R_g, \\ \sigma_2(A_g) &= \sum_{i<j} \lambda_i \lambda_j \\ &= \frac{1}{2}(|Tr\ A_g|^2 - |A_g|^2) \\ &= \frac{n}{8(n-1)} R^2 - \frac{1}{2}|Ric|^2, \\ \sigma_n(A_g) &= det(A_g) \end{aligned} \tag{4}$$

- On (M^4, g),

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- Advantages of $\sigma_2(A_g)$
(a) $R_g > 0$, $\sigma_2(A_g) > 0$, then $Ric_g > 0$ (hence $\pi_1(M)$ is finite, and $b_1 = 0$)
(b) Under conformal change of metrics,

$$A_{g_w} = A_g + \{-\nabla^2 w + dw \otimes dw - \frac{|\nabla w|^2}{2} g\}.$$

Study of symmetric functions is fully non-linear PDE of Monge-Amphere type.

$\sigma_2(Hessian)$

- Equation of Monge-Ampere type: Study

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  Dirichlet problem for u defined on $\Omega \subset \mathbb{R}^n$
- Caffarelli-Nirenberg-Spruck/ Kohn
  Krylov, Evans
  Pogorolev, Cheng-Yau, Caffarelli
  Spruck-B. Guan

$\sigma_2(Hessian)$

- Fully Non-linear PDE: for $n = 4$, $k = 2$,

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- While for A_g:

$$\begin{aligned}\sigma_2(A_{g_w})e^{4w} &= \sigma_2(A_g) + 2[(\Delta w)^2 - |\nabla^2 w|^2 \\ &+ (\nabla w, \nabla|\nabla w|^2) + \Delta w|\nabla w|^2)] \\ &+ \text{lower order terms.}\end{aligned}$$

- Definition Yamabe constant
  On (M^n, g),

$$Y(M,[g]) = inf_{g_w \in [g], vol(g_w)=1} \int_{M^n} R_{g_w} dv_{g_w}.$$

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  $$Y(M, [g]) = inf_{g_w \in [g], vol(g_w)=1} \int_{M^n} R_{g_w} dv_{g_w}.$$
- $Y(M, [g])$ is a (2nd order) conformal invariant.

Conformal invariant on M^4

- Theorem: (Chang-Gursky-Yang '02, Gursky-Viaclovsky '04)
  On (M^4, g), assume
  (i) $Y(M^4, g) > 0$;
  (ii) $\int \sigma_2(A_g) dv_g > 0$;
  then $\exists\, w \in \mathcal{C}^\infty(M)$, with $\sigma_2(A_{g_w}) \equiv 1$.

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- Part I: existence part: Under (i) and (ii) solve

$$\sigma_2(A_{g_w}) = f, \qquad \text{for some } f > 0$$

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- Part II: regularity part: Deform f to constant by method of continuity and degree theory.

Conformal Sphere Theorem

- Theorem Chang-Gursky-Yang '03
  On (M^4, g) with $Y(M^4, g) > 0$.
  (a) If
  $$\int_{M^4} \sigma_2(A_g) dv_g > \frac{1}{4} \int_{M^4} |W_g|^2 dv_g$$
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- Remark, (a) is equivalent to
  $$\int_{M^4} \sigma_2(A_g) dv_g > 4\pi^2 \chi(M^4).$$

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- (b) If
  $$\int_{M^4} \sigma_2(A_g) dv_g = \frac{1}{4} \int_{M^4} |W_g|^2 dv_g,$$
  and M^4 not diffeomorphic to S^4 or $\mathbb{R}P^4$ then either
  (1) (M^4, g) is conformal to $(\mathbb{C}P^2, g_{FS})$, or
  (2) (M^4, g) is conformal to $((S^3 \times S^1)/\Gamma, g_{prod})$.

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$$WP \equiv \frac{|W|^2 + 2|E|^2}{R^2}$$

where E denotes traceless Ricci.
(a)" $WP < \frac{1}{6}$ it turns out (a)" is equivalent to the pointwise condition (a)': $\sigma_2(A_g) > \frac{1}{4}|W|^2$
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- Margerin established previous Theorem under (a)" and (b)", while our theorem is established under the integral condition (a), (b)
- Remark: Hamilton: Assume curvature operator positive, Huisken has results for $n \geq 4$.

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  Part (a): To show under condition (a), there exists some g_w in the same conformal class of g so that the pointwise condition (a)' is satisfied, then apply Margerin's result.

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- Remark When (b) happens, the metric is a critical point of the functional $\int_{M^4} |W_g|^2 dv_g$ such metric satisfies the Bach flat condition: i.e.

$$B_{ij} = \nabla^k \nabla^l W_{ikjl} + \frac{1}{2} A^{kl} W_{ikjl} = 0.$$

  The additional PDE helps us to analyze the condition (b) to solve (b)': an additional difficulty is (b)' $\sigma_2(A_g) = \frac{1}{4}|W_g|^2$ is a degenerate PDE at the points where Weyl curvature vanishes

Finiteness Theorem

- Theorem (Chang-Qing-Yang '08) Suppose that C is a collection of Bach flat Riemannian manifolds (M^4, g) with positive Yamabe constants, satisfying
  • $\int_M |W|_g^2 dv_g \leq \Lambda_0$ for some fixed positive number Λ_0; and that
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  Then there are only finite many diffeomorphism types among manifolds in C.
- Above theorem is modeled after a result of Anderson-Cheeger '91; where finite diffeomorphism types were obtained for class of manifolds with $diam(M^n) \leq D$, $Vol(M^n) \geq v$, $|Ric| \leq \lambda$ and
$$\int_{M^n} |R_m|^{\frac{n}{2}} \leq \Lambda.$$

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- Questions:
  Q1: On (M^n, g), if $g \in {\Gamma_k}^+$, does there exist g_w with $\sigma_k(A_{g_w}) \equiv 1$?
  Q2: On what manifold does there exist a metric $g \in {\Gamma_k}^+$?

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- Q3: Above questions for metrics in negative k-cone? Trudinger-Wang-Sheng '08 local C^2 estimates fails.

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$$Y_m([g]) = inf_{g_w \in \Gamma^+_{m-1}, vol(g_w)=1} \int \sigma_m(g_w) dv_{g_w}.$$

If $Y_m > 0$ for $m \leq k$, then there exists some $g_w \in \Gamma^+_k$.

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- Localy conformally flat means locally the metric is $g = e^{2w} dx$, or equivalently Weyl curvature vanishes when $n \geq 4$.
- on (M^n, g), if $g \in \Gamma^+_k$ for some $2k \geq n$ AND locally conformally flat; then (M^n, g) is $(S^n/\Gamma, e^{2w} g_c)$.

Replacement of $\sigma_{\frac{n}{2}}(A_g)$

- We would like to seek some curvature polynomial say $v^{(2k)}(g)$ defined on M^n which satisfies (a) When g is locally conformally flat, $v^{(2k)}(g) = \sigma_k(A_g)$.
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  (b) When $2k = n$, $\int v^{(n)}(g) dv_g$ is conformally invariant.
- (c) When $2k < n$, the functional $g - - > \int v^{(2k)}(g) dv_g$ is variational in the conformal class of metric $g_w \in [g]$.

Conformal Compact Einstein manifold

- Definition: Given (X^{n+1}, M^n, g^+) with smooth boundary $\partial X = M^n$. Let r be a defining function for M^n in X^{n+1} as follows:

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r &> 0 \quad \text{in } X; \\
r &= 0 \quad \text{on } M; \\
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- (X^{n+1}, M^n, g^+) is conformally compact Einstein if g^+ is Einstein (i.e. $Ric_{g^+} = cg^+$).

Conformal Compact Einstein manifold

- Definition: Given (X^{n+1}, M^n, g^+) with smooth boundary $\partial X = M^n$. Let r be a defining function for M^n in X^{n+1} as follows:

$$\begin{aligned} r &> 0 \quad \text{in } X; \\ r &= 0 \quad \text{on } M; \\ dr &\neq 0 \quad \text{on } M. \end{aligned}$$

- We say g^+ is conformally compact, if there exists some r so that $(X^{n+1}, r^2 g^+)$ is a compact manifold.
- (X^{n+1}, M^n, g^+) is conformally compact Einstein if g^+ is Einstein (i.e. $Ric_{g^+} = cg^+$).
- We call g^+ a Poincare metric if $Ric_{g^+} = -ng^+$.

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- We can then view $(S^n, [g_c])$ as the compactification of H^{n+1} using the defining function

$$r = 2\frac{1-|y|}{1+|y|}$$

$$g_H = g^+ = r^{-2}\left(dr^2 + (1-\frac{r^2}{4})^2 g_c\right).$$

$$r^2 g^+|_{S^n} = g_c$$

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$$\begin{aligned}\mathrm{Vol}_{g^+}(\{r > \epsilon\}) &= c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots\cdots \\ &+ c_{n-1}\epsilon^{-1} + V_{g^+} + o(1)\end{aligned}$$

For n even,

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- For n odd, V_{g^+} is independent of $g \in [\bar{g}]$ where $\bar{g} = r^2 g^+|_M$, and for n even, L is independent of $g \in [\bar{g}]$, and hence are conformal invariants.

Renormalized volume–2

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- Theorem: (M. Anderson '02, Chang-Qing-Yang '06)
  On $(X^4, M^3, g+)$ conformal compact Einstein manifold, we have
  $$V_{g^+} = \frac{1}{6}\int_{X^4} \sigma_2(A_g) dv_g$$
  for any totally geodesic compactification g of g^+.
  $$8\pi^2\chi(X^4, M^3) = \int |W|_g^2 dv_g + 6V_{g^+}.$$

A class of integral invariant

- On conformal compact (X^{n+1}, M^n, g^+),

$$g^+ = \frac{1}{r^2}(dr^2 + g_r) \tag{5}$$

$$(\frac{detg_r}{detg_0})^{\frac{1}{2}} = \sum_{k \geq 0} v^{2k}(g_0) r^{2k} \tag{6}$$

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- Theorem (Chang-Fang/ Graham-Juhl)
  On (M^n, g) compact, closed; and $2k \leq n = \dim M$, define the functional
  $$\mathcal{F}_k(g) = \frac{\int_M v^{(2k)}(g) dv_g}{(\int_M dv_g)^{\frac{n-2k}{n}}},$$
  then $\mathcal{F}_g$ is variational within the conformal class; i.e., the critical metric in $[g]$ satisfies the equation
  $$v^{(2k)} = \text{constant}.$$
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- Theorem Chang-Fang-Graham
  On $(X^{n+1}, M^n, g+)$, conformally compact Einstein, n odd,
  $$V_{g^+} = c_n \int_X v^{(n+1)}(g) dv_g, \quad c_n = \frac{2^{n-1}(n+1)(\frac{n+1}{2}!)^2}{n!}$$
  for any totally geodesic compactified metric g of g^+.

A class of integral invariant

- We have

$$v^{(2)}(g) = -\frac{1}{2}\sigma_1(A_g),$$
$$v^{(4)} = \frac{1}{4}\sigma_2(A_g).$$

For $k = 6$,

$$v^{(6)}(g) = -\frac{1}{8}\left[\sigma_3(A_g) + \frac{1}{3(n-4)}(A_g)^{ij}(B_g)_{ij}\right],$$

where $(B_g)_{ij} := \frac{1}{n-3}\nabla^k\nabla^l W_{likj} + \frac{1}{n-2}R^{kl}W_{likj}$ is the Bach tensor of the metric.

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- (1) It is a "pure Ricci" curvature invariant, i.e. involves only Ricci and its derivatives.
  Obvious: when (M^n, g) is l.c.f then $v^{(2k)}(g) = (\frac{-1}{2})^k \sigma_k(A_g)$.
  Not so obvious even when $k = 6$, recall

$$v^{(6)}(g) = -\frac{1}{8}\left[\sigma_3(A_g) + \frac{1}{3(n-4)}(A_g)^{ij}(B_g)_{ij}\right],$$

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$$(B_g)_{ij} := \frac{1}{n-3}\nabla^k\nabla^l W_{likj} + \frac{1}{n-2}R^{kl}W_{likj}$$

$$\nabla^l W_{likj} = (n-3)C_{ikj} = \nabla_i(A_g)_{jk} - \nabla_k(A_g)_{ji}.$$

$$\frac{1}{n-2}R^{kl}W_{likj} = A^{kl}W_{likj} = -A_{ik};_{jk} + \nabla_i\nabla_j R - n(A^2)_{ij} + (TrA^2)g_{ij}.$$

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- (2) Under conformal change of metric $g_w = e^{2w} g$, $v^{(k)}(g_w)$ only involves second derivatives of w.

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- (2) Under conformal change of metric $g_w = e^{2w} g$, $v^{(k)}(g_w)$ only involves second derivatives of w.
- This follows from properties of Bach tensor, obstruction tensor. For example: for $v^{(6)}(g_w)$,

$$(B_g)_{ij} := \frac{1}{n-3} \nabla^k \nabla^l W_{likj} + \frac{1}{n-2} R^{kl} W_{likj}$$

$$B(g_w)_{ij} = e^{-2w}[B_{ij} - (n-4)(C_{ikj} + C_{jki}) w_k - (n-4) W_{kijl} w_k w_l]$$

This gives hope to use 2nd order elliptic PDE methods to study the $v^{(k)}$, the main difficulty is lack of ellipticity (i.e. Garding's inequality) for the equation $v^{(k)} = f$. Recent joint project with Luc Nguyen

Some open questions

- In (M^n, g) say $n = 4$, study the integral invariant $\int \sigma_2(A_g) dv_g$ when the Yamabe constant is not positive.

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- When n is odd, study M^n as boundary of conformal compact Einstein metric $(X^{n+1}, M^n, g+)$.
- Existence and Uniqueness of extension both quite open:
  • Non-uniqueness result: Hawkin-Page '83
  • Rigidity result for (S^n, g_c) Witten '81, Qing '03
  • Existence result on neighborhood of (S^n, g_c) Graham-Lee
  • regularity result: Smoothness of (M^n, g) implies smoothness of compactified metrics (Helliwell '06 (n=3) and Chruscial-Delay-Lee-Skinner (general n)).
  • Compactness result: Does the compactness of $(M^n, [g])$ determines the compactness of the class of compactified metrics on X^{n+1} ? Chang-Yang '11

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- (3) $\frac{1}{C_4} \leq Y(\partial X, [g_+^\alpha]) < Y(S^3, [g_c])$,
- (4) $S(g_+^\alpha) = 0$.

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- (a) the Yamabe metrics $\{(X, g_Y^\alpha)\}$ with $Vol(g_Y^\alpha) = 1$ is C^∞ compact.
- (b) a subsequence of the dilated Yamabe metrics converges in Gromov-Hausdroff sense to an asymptotically flat conformally Einstein metric $(X_\infty, \partial X_\infty, g_+^\infty)$ with boundary, which has a smooth conformal compactification, and the conformal infinity h of the compactification is a limit of the sequence h^α.

A boundary conformal invariant S

- The 2-tensor S on the boundary which is pointwisely conformally invariant. The definition of S is motivated by the Gauss-Bonnet formula on 4-manifolds with boundary $(X, \partial X, g)$, with g defined on X and extended smoothly to ∂X. Consider the functional on $(X, \partial X, g)$,

$$g \rightarrow \int_X |W|_g{}^2 dv_g + 8 \int_{\partial X} W_{\alpha n \beta n} L^{\alpha\beta} d\sigma_g, \tag{7}$$

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- Critical metrics of the functional satisfy in the interior the well-known condition $B_{ij} = 0$ where B_{ij} is the Bach tensor, and on the boundary

$$0 = S_{\alpha\beta} = \nabla^i W_{i\alpha n\beta} + \nabla^i W_{i\beta n\alpha} - \nabla^n W_{n\alpha n\beta} + \frac{4}{3} H W_{n\alpha n\beta}.$$

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- Recent joint effort of Chang-Ge-Yang, replace the Yamabe metric by other metrics and replace the $S = 0$ condition by less restrictive condition.