CHAPTER 12
INTEGRATION ON $\mathbb{R}^n$

12.1 JORDAN REGIONS

DEFINITION. A grid on a rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ is a collection of rectangles $\mathcal{G}$ defined in following way. For $j = 1, \cdots, n$ $\mathcal{P}_j(\mathcal{G}) = \{x^{(j)}_k : k = 1, \cdots, n_j\}$ is a partition of $[a_j, b_j]$, then $\mathcal{G}$ is the collection of rectangles of the form $I_1 \times \cdots \times I_n$ where $I_j = [x^{(j)}_{k-1}, x^{(j)}_k]$ for $j = 1, \cdots, n$. [Grid is partition in $\mathbb{R}^n$]. A grid $\mathcal{H}$ is said finer than $\mathcal{G}$ if and only $\mathcal{P}_j(\mathcal{G}) \subset \mathcal{P}_j(\mathcal{H})$ for $j = 1, \cdots, n$.

DEFINITION. The outer sum of $E$ with respect to a grid $\mathcal{G}$ of the rectangle $R$ is

$$ V(E, \mathcal{G}) := \sum_{R_j \cap E \neq \emptyset} |R_j|. $$

Remark. Let $R$ be a $n$-dimensional rectangle.

(1) Let $E$ be a subset of $R$, and let $\mathcal{G}, \mathcal{H}$ be grids on $R$. If $\mathcal{G}$ is finer than $\mathcal{H}$ then

$$ V(E, \mathcal{G}) \leq V(E, \mathcal{H}). $$

(2) If $A \subset B \subset R$, then $V(A, \mathcal{G}) \leq V(B, \mathcal{G})$.

Example. Let $R = [0, 1] \times [0, 1]$ and let $A = R \cap \mathbb{Q}^2, B = R \setminus A$, then $V(A, \mathcal{G}) = V(B, \mathcal{G}) = V(R, \mathcal{G})$.

DEFINITION. A subset $E$ of $\mathbb{R}^n$ is Jordan region if and only if there is a rectangle $R$ contains $E$, and for each $\epsilon > 0$, there is a grid on $R$ such that $V(\partial, \mathcal{G}) < \epsilon$.

DEFINITION. Let $E$ be a Jordan region in $\mathbb{R}^n$ and $R$ is a rectangle in $\mathbb{R}^n$ contains $E$. The volume( or the Jordan content ) of $E$ is defined by

$$ Vol(E) = \inf_{\mathcal{G}} V(E, \mathcal{G}). $$

Remark. $Vol(E)$ is independent of the choice of $R$.

Remark. For $R$ is a rectangle of $\mathbb{R}^n$, then $Vol(R) = |R|$.

Remark. Suppose that $E$ is a bounded subset in $\mathbb{R}^n$.

(1) $E$ is a Jordan region of volume zero if and only if for each $\epsilon > 0$ we can find a grid $\mathcal{G}$ such that $V(E, \mathcal{G}) < \epsilon$.

(2) $E$ is a Jordan region if and only if $Vol(\partial E) = 0$.

(3) If $E$ is a set of volume zero and $A \subset E$, then $A$ is a Jordan region and $Vol(A) = 0$. 

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DEFINITION. Let $\mathcal{E} = \{E_j\}$ be a collection of subsets of $\mathbb{R}^n$.

1. $\mathcal{E}$ is said to be nonoverlapping if and only if $E_j \cap E_k$ is of volume zero for $j \neq k$.
2. $\mathcal{E}$ is said to be pairwise disjoint if and only if $E_j \cap E_k = \emptyset$ is for $j \neq k$.

Theorem. Let $E$ be a subset of $\mathbb{R}^n$. Then $E$ is a Jordan region of volume zero if and only if for every $\epsilon > 0$ there is a finite collection of cubes $Q_k$ of the same size such that

$$\overline{E} \subset \bigcup_1^q Q_k \text{ and } \sum_1^q |Q_k| < \epsilon.$$ 

Corollary. If $E_1, E_2$ are Jordan regions, then $E_1 \cup E_2$ is a Jordan region and

$$\text{Vol}(E_1 \cup E_2) \leq \text{Vol}(E_1) + \text{Vol}(E_2).$$

Corollary. Suppose that $V$ is a bounded open set in $\mathbb{R}^n$ and $\phi : V \to \mathbb{R}$ is 1-1 and $C^1$ on $V$ with $\Delta + \phi \neq 0$.

1. If $E$ is of volume zero and $\overline{E} \subset V$, then $\phi(E)$ is of volume zero.
2. If $\{E_k\}, k \in \mathbb{N}$ is a nonoverlapping collection of subsets in $\mathbb{R}^n$ with $E_k \subset V$ for all $k \in \mathbb{N}$, then $\phi(E = k)$ is a nonoverlapping collection of sets in $\mathbb{R}^n$.
3. If $E$ is a Jordan region and $\overline{E} \subset V$, then $\phi(E)$ is a Jordan region.

12.2 RIEMANN INTEGRATION ON JORDAN REGION

DEFINITION. Let $E$ be a Jordan region in $\mathbb{R}^n$, let $f : E \to \mathbb{R}$ is a bounded function, let $R$ be a rectangle such that $E \subset R$ and let $\mathcal{G} = \{R_1, \cdots, R_p\}$ be a grid on $R$. Extend $f$ to $\mathbb{R}^n$ by setting $f(x) = 0$ for $x \in \mathbb{R}^n \setminus E$.

1. The upper sum of $f$ on $E$ with respect to $\mathcal{G}$ is

$$U(f, \mathcal{G}) = \sum_{R_j \cap E \neq \emptyset} M_j |R_j|$$

where $M_j = \sup_{x \in R_j} f(x)$.
2. The lower sum of $f$ on $E$ with respect to $\mathcal{G}$ is

$$L(f, \mathcal{G}) = \sum_{R_j \cap E \neq \emptyset} m_j |R_j|$$

where $m_j = \inf_{x \in R_j} f(x)$.
3. The upper and lower integrals of $f$ on $E$ are defined by

$$(U) \int_E f(x) dx := (U) \int_E f dV := \inf_{\mathcal{G}} U(f, \mathcal{G})$$

and

$$(L) \int_E f(x) dx := (L) \int_E f dV := \sup_{\mathcal{G}} L(f, \mathcal{G}),$$

where the supremum and infimum are taken over all grids on $R$. 
Remark. Let \( E \) be a nonempty Jordan region in \( \mathbb{R}^n \), let \( f : E \to \mathbb{R} \) be bounded and let \( R \) be a rectangle that contains \( E \).

1. If \( \mathcal{G} \) and \( \mathcal{H} \) are grids on \( R \), then \( L(f, \mathcal{G}) \leq U(f, \mathcal{H}) \).
2. The upper and lower integrals of \( f \) over \( E \) exist, do not depend on the choice of \( R \), and satisfy

\[
(L) \int_E f \, dV \leq (U) \int_E f \, dV.
\]

DEFINITION. A real-valued bounded function \( f \) defined on a Jordan region \( E \) is said to be (Riemann) integrable on \( E \) if and only if for every \( \varepsilon > 0 \) there is a grid \( \mathcal{G} \) such that

\[
U(f, \mathcal{G}) - L(f, \mathcal{G}) < \varepsilon.
\]

Remark. Let \( E \) be a Jordan region in \( \mathbb{R}^n \) and suppose that \( f : E \to \mathbb{R} \) is bounded. Then \( f \) is integrable on \( E \) if and only if

\[
(U) \int_E f(x) \, dx = (L) \int_E f(x) \, dx.
\]

When \( f \) is integrable on \( E \) we denote the common value by \( \int_E f(x) \, dx \) or \( \int_E f \, dV \).

**Theorem.** Let \( E \) be a Jordan region in \( \mathbb{R}^n \) and let \( R \) be a rectangle that contains \( E \), and suppose that \( f : E \to \mathbb{R} \) is integrable on \( E \). If \( g(x) = f(x) \) for \( x \in E \) and \( g(x) = 0 \) for \( x \in \mathbb{R}^n \setminus E \), then \( g \) is integrable on \( R \) and \( \int_R g \, dV = \int_E f \, dV \).

**Theorem.** Let \( E \) be a Jordan region in \( \mathbb{R}^n \) and suppose that \( f : E \to \mathbb{R} \) is bounded. Then given \( \varepsilon > 0 \) there is a grid \( \mathcal{G}_0 \) such that if \( \mathcal{G} = \{R_1, \ldots, R_p\} \) is any grid finer than \( \mathcal{G}_0 \), then

\[
| (U) \int_E f(x) \, dx - \sum_{R_j \in \mathcal{G}_0} M_j |R_j|| < \varepsilon
\]

and

\[
| (L) \int_E f(x) \, dx - \sum_{R_j \in \mathcal{G}_0} m_j |R_j|| < \varepsilon.
\]

**Theorem.** If \( E \) is a closed Jordan region in \( \mathbb{R}^n \) and \( f : E \to \mathbb{R} \) is continuous on \( E \), then \( f \) is integrable on \( E \).

**Theorem.** If \( E \) is a closed Jordan region, then

\[
Vol(E) = \int_E 1 \, dV.
\]
Theorem. [LINEAR PROPERTIES]. Let $E$ be a Jordan region in $\mathbb{R}^n$ and suppose that $f, g : E \to \mathbb{R}$, and let $\alpha$ be a scalar.

(1) If $f, g$ are integrable on $E$, then so are $\alpha f$ and $f + g$. In fact
\[
\int_E \alpha f(x) \, dx = \alpha \int_E f(x) \, dx
\]
and
\[
\int_E (f(x) + g(x)) \, dx = \int_E f(x) \, dx + \int_E g(x) \, dx.
\]

(2) If $E_1, E_2$ are nonoverlapping Jordan regions and $f$ is integrable on both $E_1$ and $E_2$, then $f$ is integrable on $E_1 \cup E_2$ and
\[
\int_{E_1 \cup E_2} f(x) \, dx = \int_{E_1} f(x) \, dx + \int_{E_2} f(x) \, dx.
\]

Theorem. Let $E$ be a Jordan region in $\mathbb{R}^n$ and suppose that $f, g : E \to \mathbb{R}$ are bounded functions.

(1) If $E_0$ is of volume zero, then $f$ is integrable on $E_0$ and
\[
\int_{E_0} f(x) \, dx = 0.
\]

(2) If $f$ is integrable on $E$ and there is a subset $E_0$ of $E$ such that $Vol(E_0) = 0$ and $f(x) = g(x)$ for all $x \in E \setminus E_0$, then $g$ is integrable on $E$ and
\[
\int_{E} g(x) \, dx = \int_{E} f(x) \, dx.
\]

Theorem. [COMPARISON THEOREM]. Let $E$ be a Jordan region in $\mathbb{R}^n$ and suppose that $f, g : E \to \mathbb{R}$ are integrable on $E$.

(1) If $f(x) \leq g(x)$ on $E$, then
\[
\int_{E} f(x) \, dx \leq \int_{E} g(x) \, dx.
\]

(2) If $M, m$ are scalar satisfy $m \leq f(x) \leq M$ for $x \in E$, then
\[
mVol(E) \leq \int_{E} f(x) \, dx \leq MVol(E).
\]

(3) The function $|f|$ is integrable on $E$ and
\[
| \int_{E} f(x) \, dx | \leq \int_{E} |f(x)| \, dx.
\]
**Theorem.** [MEAN VALUE THEOREM]. Let $E$ be a Jordan region in $\mathbb{R}^n$ and suppose that $f, g : E \to \mathbb{R}$ are integrable on $E$ with $g(x) \geq 0$ for all $x \in E$.

(1) Then there is a number $c$ satisfying $\inf_{x \in E} f(x) \leq c \leq \sup_{x \in E} f(x)$ such that

$$c \int_E f(x) \, dx = \int_E f(x) g(x) \, dx.$$ 

(2) There is a number $c$ satisfying $\inf_{x \in E} f(x) \leq c \leq \sup_{x \in E} f(x)$ such that

$$c \Vol(E) = \int_E f(x) \, dx.$$

**DEFINITION.** A set $E \subset \mathbb{R}^n$ is said to be of measure zero if and only if for every $\epsilon > 0$ there is a countable collection of rectangles $\{R_j\}$ such that

$$E \subset \bigcup_{1}^{\infty} R_j \text{ and } \sum_{1}^{\infty} |r_j| < \epsilon.$$

**Remark.** If $\{E_j\}$ be a sequence of measure zero subset of $\mathbb{R}^n$, then $E = \bigcup_{1}^{\infty} E_j$ is also of measure zero.

**Theorem.** Let $E$ be a Jordan region in $\mathbb{R}^n$ and suppose that $f : E \to \mathbb{R}$ is bounded.

(1) Then $f$ is Riemann integrable on $E$ if and only if the set of points of discontinuity of $f$ is of measure zero.

(2) Suppose that $V$ is an open set in $\mathbb{R}^n$ such that $\overline{E} \subset V$, and $\phi : V \to \mathbb{R}^n$ is 1-1 and $\phi^{-1}$ is $C^1$ on $\phi(V)$ with $\Delta_{\phi^{-1}} \neq 0$. If $f$ is integrable on $\phi(E)$, then $f \circ \phi$ is integrable on $E$.

**12.3 ITERATED INTEGRALS**

**Lemma.** Let $R = [a, b] \times [c, d]$ be a two dimensional rectangle and suppose that $f : R \to \mathbb{R}$ is bounded. If $f(x \cdot)$ is integrable on $[c, d]$ for each $x \in [a, b]$, then

$$(L) \int_R f \, dA \leq (L) \int_{a}^{b} \left( \int_{c}^{d} f(x, y) \, dy \right) dx$$

$$\leq (U) \int_{a}^{b} \left( \int_{c}^{d} f(x, y) \, dy \right) dx \leq (U) \int_R f \, dA.$$
Theorem. [FUBINI’S THEOREM] Let \( R = [a, b] \times [c, d] \) be a rectangle and let \( f : R \to \mathbb{R} \). Suppose that \( f(x, \cdot) \) is integrable on \([c, d]\) for each \( x \in [a, b] \), \( f(\cdot, y) \) is integrable on \([a, b]\) for each \( y \in [c, d] \), and that \( f \) is integrable on \( R \) (as a function in two variables), then

\[
\iint_R f \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy.
\]

Example.

\[
\int_0^1 \int_0^1 y^3 e^{xy^2} \, dy \, dx.
\]

Remark. Let \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) be defined by

\[
f(x, y) = \begin{cases} 
2^{2n} & \text{for } (x, y) \in [2^{-n}, 2^{-n+1}) \times [2^{-n}, 2^{-n+1}), \\
-2^{2n+1} & \text{for } (x, y) \in [2^{-n-1}, 2^{-n}) \times [2^{-n}, 2^{-n+1}), \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( \int_0^1 \int_0^1 f(x, y) \, dx \, dy = 0 \) but \( \int_0^1 \int_0^1 f(x, y) \, dy \, dx = 1 \).

Remark. Let \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) be defined by \( f(x, y) = 1 \) if \( (x, y) = (\frac{p}{2^n}, \frac{q}{2^n}) \) and \( f(x, y) = 0 \) otherwise. Then

\[
\int_0^1 \int_0^1 f(x, y) \, dx \, dy = 0 = \int_0^1 \int_0^1 f(x, y) \, dy \, dx,
\]

but \( f \) is not integrable on \( R \).

Remark. Let \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) be defined by \( f(x, y) = 0 \) for \( x = 0 \) or when \( x \) or \( y \) is irrational, and \( f(x, y) = 1/q \) when \( x, y \in \mathbb{Q} \) and \( x = p/q \) in reduced form. Then \( f \) is integrable on \( R \) and \( f(\cdot, y) \) is integrable on \([0, 1] \) for all \( y \in [0, 1] \) but \( f(x, \cdot) \) is not integrable on \([0, 1] \) for infinite many \( x \in [0, 1] \).

Lemma. Let \( R = [a_1, b_1] \times \cdots \times [a_n, b_n] \) be an \( n \)-rectangle and let \( f : R \to \mathbb{R} \) be integrable on \( R \). If for each \( x = (x_1, \cdots, x_{n-1}) \in R_n \) \( : = [a_1, b_1] \times \cdots \times [a_{n-1}, b_{n-1}] \), the function \( f(x, \cdot) \) is integrable on \([a_n, b_n] \), then \( \int_{a_n}^{b_n} f(x, t) \, dt \) is integrable on \( R_n \) and

\[
\int_R f(x, t) \, d(x, t) = \int_{R_n} \int_{a_n}^{b_n} f(x, t) \, dt \, dx.
\]

DEFINITION. \( E \subset \mathbb{R}^n \) is a projectable region if and only if there is a closed Jordan region in \( H \subset \mathbb{R}^{n-1} \) and index \( j \in \{1, \cdots, n\} \), and continuous functions \( \phi, \psi : H \to \mathbb{R} \) such that

\[
E = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : (x_1, \cdots, \hat{x}_j, \cdots, x_n) \in H \}
\]

such that

\[
\phi(x_1, \cdots, \hat{x}_j, \cdots, x_n) \leq x_j \leq \psi(x_1, \cdots, \hat{x}_j, \cdots, x_n).
\]

DEFINITION. \( \Pi_k = \{x \in \mathbb{R}^n : x_k = 0\} \).
Theorem. Let $E$ be a projectable region in $\mathbb{R}^n$ generated by $j, \phi, \psi$ and $H$. Then $E$ is a Jordan region in $\mathbb{R}^n$. Moreover, if $f : E \to \mathbb{R}$ is continuous on $E$, then

$$\int_E f(x)dx = \int_H (\int_{\phi(x_1, \ldots, x_j, \ldots, x_n)} f(x_1, \ldots, x_n)dx_j)d(x_1, \ldots, x_j, \ldots, x_n).$$

Example. $E$ is bounded by $x + y + z = 1, x = 0, y = 0, z = 0$ and $f(x, y, z) = x$.

Example. $E$ is bounded by $|x| = 1, z = x^2 - y^2$, where $z \geq 0$, and $f(x, y, z) = x^2$.

Example. $E$ is bounded by $z = y^2, z = 1, z = x, x = 0$ and $f(x, y, z) = x - z$.

12.4 CHANGE OF VARIABLES

Lemma. Let $W$ be open in $\mathbb{R}^n$, let $\phi : W \to \mathbb{R}^n$ be 1-1 and continuously differentiable on $W$ with $\Delta_\phi \neq 0$ on $W$, and suppose that $\phi^{-1}$ is continuously differentiable on $\phi(W)$ with $\Delta_{\phi^{-1}} \neq 0$ on $\phi(W)$. Suppose further that $|R| = \int_{\phi^{-1}(R)} |\Delta_\phi(x)|dx$ for every $n$-dimensional rectangle $R \subset \phi(W)$. If $E$ is a Jordan region with $\overline{E} \subset W$, if $f$ is integrable on $\phi(E)$, and if $f \circ \phi$ is integrable on $E$, then

$$\int_{\phi(E)} f(u)du = \int_E (f \circ \phi)(x)|\Delta_\phi(x)|dx.$$

Lemma. Let $V$ be open in $\mathbb{R}^n$, let $\phi : V \to \mathbb{R}^n$ be 1-1 and continuously differentiable on $V$. If $\Delta_\phi(a) \neq 0$ for some $a \in V$, then there is an open rectangle $W$ such that $a \in W \subset V$, $\Delta_\phi$ is nonzero on $W, \phi^{-1}$ is $C^1$, and its Jacobian is nonzero on $\phi(W)$, and such that if $R$ is an $n$-rectangle contained in $\phi(W)$, then $\phi^{-1}(R)$ is Jordan and

$$|R| = \int_{\phi^{-1}(R)} |\Delta_\phi(x)|dx.$$

Lemma. Suppose that $V$ is an open set in $\mathbb{R}^n, a \in V$ and $\phi : V \to \mathbb{R}^n$ is continuously differentiable on $V$. If $\Delta_\phi(a) \neq 0$, then there exists an open rectangle $W \subset V$ containing $a$ such that if $E$ is Jordan with $\overline{E} \subset W$, if $f \circ \phi$ is integrable on $E$, and if $f$ is integrable on $\phi(E)$, then

$$\int_{\phi(E)} f(u)du = \int_E (f(\phi(x))|\Delta_\phi(x)|dx.$$
Theorem. Suppose that $V$ is an open set in $\mathbb{R}^n$ and $\phi : V \to \mathbb{R}^n$ is 1-1 and continuously differentiable on $V$. If $\Delta_\phi \neq 0$ on $V$, if $f \circ \phi$ is integrable on $E$, and if $f$ is integrable on $\phi(E)$, then

$$
\int_{\phi(E)} f(u) du = \int_E (f(\phi(x)))|\Delta_\phi(x)| dx.
$$

Example. Find the volume bounded by $z = x^2 + y^2, x^2 + y^2 = 4, z = 0$.

Example. Let $E = \{(x, y) : a^2 \leq x^2 + y^2 \leq 1$ and $0 \leq y \leq x\}$, find

$$
\iint_E \frac{x^2 + y^2}{x} dA.
$$

Example. Find the volume of the region $E$ that lies inside $x^2 + y^2 + z = 4$, outside $x^2 - 2x + y^2 = 0$ and above $z = 0$.

Example. $\iint_Q x dW$ where $Q = B_3(0, 0, 0) \setminus B_2(0, 0, 0)$.

Example. $\iint_E \sin(x+y) \cos(2x-y) dA$, where $E$ is bounded by $y = 2x - 1, y = 2x + 3, y = -x, y = -x + 1$. 