3. The $L^p$ Space

In this section we consider a space $L^p(E)$ which resembles $\ell^p$ on many aspects. After general concepts of measure and integral were introduced, we will see that these two spaces can be viewed as special cases of a more general $L^p$ space.

**Definition 3.1.** Given a measurable set $E \subset \mathbb{R}^n$. For $0 < p < \infty$, define the space $L^p(E)$ and the real-valued function $\| \cdot \|_p$ on $L^p(E)$ by

$$L^p(E) = \{ f : f \text{ is measurable on } E \text{ and } \int_E |f|^p < \infty \}, \quad \|f\|_p = \left( \int_E |f|^p \right)^{\frac{1}{p}}.$$

The *essential supremum* of a measurable function $f$ on $E$ is defined by

$$\text{ess sup } f = \inf_{E} \{ \alpha \in (-\infty, \infty) : m(\{ f > \alpha \}) = 0 \}.$$

The space $L^\infty(E)$ and the real-valued function $\| \cdot \|_\infty$ on $L^\infty(E)$ are given by

$$L^\infty(E) = \{ f : f \text{ is measurable on } E \text{ and } \text{ess sup}_{E} |f| < \infty \}, \quad \|f\|_\infty = \text{ess sup}_{E} |f|.$$

Functions in $L^\infty(E)$ are said to be *essentially bounded*.

The measurable function $f$ in the definition of $L^p(E)$ for $0 < p < \infty$ can be complex-valued, but functions in $L^\infty(E)$ are assumed to be real-valued. We leave it to the readers to check that $m(f > \text{ess sup}_E f) = 0$ for any $f \in L^\infty(E)$ (Exercise 3.1). In other words, $f \leq \text{ess sup}_E f$ and $|f| \leq \|f\|_\infty$ almost everywhere.

For any $0 < p \leq \infty$, two functions $f_1, f_2$ in $L^p(E)$ are considered equivalent if $f_1 = f_2$ almost everywhere on $E$. The space of equivalence classes, still denoted by $L^p(E)$, are called $L^p(E)$ classes or $L^p(E)$ spaces.

Similar to $\ell^p$, the space $L^p(E)$ is a vector space for any $0 < p \leq \infty$. Indeed, $\|\alpha f\|_p = |\alpha| \|f\|_p$ for any scalar $\alpha$, $\|\alpha f\|_p = \|\alpha\|_p \|f\|_p$ and

$$f,g \in L^p(E) \Rightarrow |f + g|^p \leq (2 \max\{|f|, |g|\})^p \leq 2^p(|f|^p + |g|^p),$$

$$f,g \in L^{\infty}(E) \Rightarrow \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

The second line follows by observing that

$$\begin{cases} 
|f| \leq \|f\|_\infty \text{ almost everywhere} \\
|g| \leq \|g\|_\infty \text{ almost everywhere} \end{cases} \Rightarrow |f + g| \leq \|f\|_\infty + \|g\|_\infty \text{ almost everywhere}.$$

When $1 \leq p \leq \infty$, the function $\| \cdot \|_p$ is a norm on $L^p(E)$. This follows from the theorem below, the proof for which is similar to that of $\ell^p$.

**Theorem 3.1.** Given $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $f, g$ be measurable functions on $E \subset \mathbb{R}^n$.

(a) **Hölder’s Inequality for $L^p$** If $f \in L^p(E)$, $g \in L^q(E)$, then $fg \in L^1(E)$ and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

(b) **Minkowski’s Inequality for $L^p$**

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$
Proof. (a) The cases $p = 1$, $q = \infty$ and $p = \infty$, $q = 1$ are obvious. Consider $1 < p, q < \infty$. If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $fg = 0$ almost everywhere on $E$, and the asserted inequality is obvious. We may now assume $0 < \|f\|_p$, $\|g\|_q < \infty$.

Let $F = \frac{f}{\|f\|_p}$, $G = \frac{g}{\|g\|_q}$. By Young’s inequality,

$$
\int_E |FG| \leq \int_E \frac{|F|^p}{p} + \frac{|G|^q}{q} = \frac{\|f\|^p}{p} + \frac{\|g\|^q}{q} = \frac{1}{p} + \frac{1}{q} = 1,
$$

$$
\|fg\|_1 = \int_E |fg| = \|f\|_p \|g\|_q \int_E |FG| \leq \|f\|_p \|g\|_q.
$$

(b) The case $p = 1$ is obvious, and the case $p = \infty$ has been proved. Now we consider $1 < p < \infty$. Note that $q = \frac{p}{p-1}$. Minkowski’s inequality follows easily from (a):

$$
\|f + g\|_p^p = \int_E |f + g|^p \leq \int_E |f + g|^{p-1}|f| + \int_E |f + g|^{p-1}|g| = \left(\int_E |f + g|^p\right)^{p-1} \left(\int_E |f|^p\right)^{\frac{1}{p}} + \left(\int_E |f + g|^p\right)^{p-1} \left(\int_E |g|^p\right)^{\frac{1}{p}} = \|f + g\|_p^{p-1} (\|f\|_p + \|g\|_p).
$$

The special case $p = q = 2$ of the Hölder inequality is also known as the Cauchy-Schwarz inequality. The assumption $1 \leq p \leq \infty$ is necessary. For example, let $E = [0, 1]$, $f = \chi_{[0, \frac{1}{2}]}$, $g = \chi_{[\frac{1}{2}, 1]}$. Then

$$
\|f\|_p + \|g\|_p = \left(\frac{1}{2}\right)^{\frac{1}{p}} + \left(\frac{1}{2}\right)^{\frac{1}{p}} = 2^{1 - \frac{1}{p}} < 1 = \|f + g\|_p.
$$

Corollary 3.2. Suppose $0 < p < q < \infty$, $m(E) < \infty$. Then

$$
\left(\frac{1}{m(E)} \int_E |f|^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{m(E)} \int_E |f|^q\right)^{\frac{1}{q}}.
$$

In particular, $L^q(E) \subset L^p(E)$.

Proof. Let $r = \frac{q}{q-p}$, then $\frac{1}{q/p} + \frac{1}{r} = 1$. Therefore,

$$
\int_E |f|^p \leq \left(\int_E (|f|^p)^{\frac{r}{q}}\right)^{\frac{q}{r}} \left(\int_E 1^r\right)^{\frac{1}{r}} = \left(\int_E |f|^q\right)^{\frac{r}{q}} m(E)^{\frac{q-p}{q}}.
$$

Then the corollary follows from

$$
\|f\|_p = \left(\int_E |f|^p\right)^{\frac{1}{p}} \leq \|f\|_q m(E)^{\frac{q-p}{q}} = \|f\|_q m(E)^{\frac{1}{p} - \frac{1}{q}}.
$$

\[\square\]
EXAMPLE 3.1. Consider \( f(x) = x^r, \ r \neq 0 \), defined on \([0, \infty)\).

When \( r < 0 \), \( f \in L^p([1, \infty]) \) if and only if \( p > -\frac{1}{r} \), \( f \in L^p(0, 1) \) if and only if \( 0 < p < -\frac{1}{r} \).

When \( r > 0 \), \( f \notin L^p([1, \infty]) \) for any \( p > 0 \), \( f \in L^p(0, 1) \) for any \( p > 0 \).

This example shows that the assumption \( m(E) < \infty \) is necessary in the above corollary, and \( L^q(E) \subseteq L^p(E) \) if \( 0 < p < q < \infty \) and \( E = [1, \infty) \).

EXAMPLE 3.2. The function \( \log x \) belongs to \( L^p(0, 1) \) for any \( 0 < p < \infty \) but it is not in \( L^{\infty}(0, 1) \).

THEOREM 3.3. (Riesz-Fisher) For any \( 1 \leq p \leq \infty \), the space \((L^p(E), \| \cdot \|_p)\) is a Banach space.

PROOF. Consider \( p = \infty \) first. Note that convergence in \( L^{\infty}(E) \) means uniform convergence outside a set of measure zero.

Let \( \{f_n\} \) be a Cauchy sequence in \( L^{\infty}(E) \). For each \( n, m \in \mathbb{N} \), \( |f_n - f_m| \leq \|f_n - f_m\|_{\infty} \) except on a set \( Z_{n,m} \) of measure zero. Let \( Z = \bigcup_{n,m \in \mathbb{N}} Z_{n,m} \), then \( Z \) has measure zero and

\[ |f_n - f_m| \leq \|f_n - f_m\|_{\infty} \quad \text{on } E \setminus Z \]

In particular, for any \( x \in E \setminus Z \), \( \{f_n(x)\} \) converges. Let \( f(x) = \lim_{n \to \infty} f_n(x) \) for \( x \in E \setminus Z \) and set \( f(x) = 0 \) on \( Z \). Then \( f_n \to f \) uniformly on \( E \setminus Z \).

This implies that \( f_n \) converges to \( f \) in \( L^{\infty}(E) \), and so \( L^{\infty}(E) \) is complete.

Now we consider \( 1 \leq p < \infty \). By Theorem 1.3, we only have to show that every absolutely convergent series converges to some element in \( L^p(E) \).

Let \( \sum_{k=1}^{\infty} f_k \) be an absolutely convergent series. Then \( \sum_{k=1}^{\infty} \|f_k\|_p = M \) is finite. Let

\[ g_n = \sum_{k=1}^{n} |f_k|, \quad s_n = \sum_{k=1}^{n} f_k. \]

By Minkowski’s inequality, \( \|g_n\|_p \leq \sum_{k=1}^{n} \|f_k\|_p \leq M \). Thus \( \int_E g_n^p \leq M^p \) for any \( n \). For any \( x \in E \), the function \( g_n(x) \) is increasing in \( n \), and so \( g_n \) converges pointwise to some function \( g : E \to [0, \infty] \).

The function \( g \) is measurable and, by Fatou’s lemma,

\[ \int_E g^p \leq \liminf_{n \to \infty} \int_E g_n^p \leq M^p. \]

Therefore \( g \) is finite almost everywhere and \( g \in L^p(E) \). When \( g(x) \) is finite, \( \sum_{k=1}^{\infty} f_k(x) \) is absolutely convergent. Let \( s(x) \) be its value, and set \( s(x) = 0 \) elsewhere. Then the function \( s \) is defined everywhere, measurable on \( E \), and

\[ \sum_{k=1}^{n} f_k = s_n \to s \quad \text{almost everywhere on } E. \]

Since \( |s_n(x)| \leq g(x) \) for all \( n \), we have \( |s(x)| \leq g(x) \), where hence \( s \in L^p(E) \) and \( |s_n(x) - s(x)| \leq 2g(x) \in L^p(E) \). By the Lebesgue dominated convergence theorem,

\[ \int_E |s_n - s|^p \to 0 \quad \text{as } n \to \infty. \]

This proves that \( \sum_{k=1}^{\infty} f_k \) converges to \( s \in L^p(E) \), and thus proves completeness of \( L^p(E) \). \( \square \)
Theorem 3.4. If $1 \leq p < \infty$, then $L^p(E)$ is separable.

Proof. Consider $E = \mathbb{R}^n$. Consider the collection of cubes of the form $[k_1, k_1 + 1] \times \ldots \times [k_n, k_n + 1]$, $k_1, \ldots, k_n \in \mathbb{Z}$. Bisect each of these cubes into $2^n$ congruent subcubes, and repeat this process. The collection of all these cubes is called dyadic cubes. Let $\mathcal{D}$ be the set of finite linear combinations of characteristic functions on these dyadic cubes with rational coefficients. Clearly $\mathcal{D}$ is countable. All we need to prove is that $\mathcal{D}$ is dense in $L^p(\mathbb{R}^n)$. That is, given $f \in L^p(\mathbb{R}^n)$, there exists a sequence $f_k \in \mathcal{D}$ such that $\|f_k - f\|_p \to 0$ as $k \to \infty$.

It suffices to consider the case $f \geq 0$ since

$$f = f^+ - f^-, \quad \|f_k - f\|_p \leq \|f_k^+ - f^+\|_p + \|f_k^- - f^-\|_p$$

(by Minkowski’s inequality). In fact, it suffices to consider the case $f \geq 0$ with compact support since

$$\int_{\mathbb{R}^n} |f_k - f|^p = \lim_{m \to \infty} \int_{[-m,m]^n} |f_k - f|^p$$

Let $\{g_k\}$ be an increasing sequence of nonnegative simple functions such that $g_k \nearrow f$, $f \geq 0$ has compact support. Then, by the monotone convergence theorem,

$$g_k \in L^p(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} |g_k - f|^p \to 0 \quad \text{as } k \to \infty.$$ 

Therefore, it suffices to consider the case $f \geq 0$, $f$ is a simple function with compact support.

For a simple function $f = \sum_{k=1}^{N} a_k \chi_{E_k}$,

$$\int_{\mathbb{R}^n} |f - g|^p = \sum_{k=1}^{N} \int_{E_k} |a_k - g|^p \quad \text{for any } g \in L^p(\mathbb{R}^n).$$

From this observation, it suffices to consider the case when $f$ is the characteristic function of some bounded measurable set $E$. There exists a $G_\delta$ set $G$ containing $E$ with $m(G \setminus E) = 0$, so that we may consider only the case $E$ being a $G_\delta$ set.

Let $E = \bigcap_{k=1}^{\infty} \mathcal{O}_k$, $\mathcal{O}_1 \supset \mathcal{O}_2 \supset \cdots$ is a nested sequence of bounded open sets. Then, by the monotone convergence theorem,

$$\int_{\mathbb{R}^n} |\chi_{\mathcal{O}_k} - \chi_E|^p \to 0 \quad \text{as } k \to \infty.$$ 

Therefore, it suffices to consider $f = \chi_{\mathcal{O}}$, where $\mathcal{O}$ is a bounded open set. But in this case, $f = \sum_{k=1}^{\infty} \chi_{c_k}$ for some dyadic cubes $c_k$. This proves $\mathcal{D}$ is dense in $L^p(\mathbb{R}^n)$.

Now consider arbitrary measurable set $E \subset \mathbb{R}^n$. Let $\mathcal{D}' = \{g \cdot \chi_E : g \in \mathcal{D}\}$. Then $\mathcal{D}'$ is a countable set consisting of finite linear combinations of characteristic functions on dyadic cubes which intersect with $E$ and with rational coefficients.

Given $f \in L^p(E)$. Let $\tilde{f} = f$ on $E$, $\tilde{f} = 0$ on $\mathbb{R}^n \setminus E$. Choose $\{f_k\} \subset \mathcal{D}$ such that $\int_{\mathbb{R}^n} |f_k - \tilde{f}|^p \to 0$ as $k \to \infty$. Then

$$\int_{E} |f_k - \chi_E - \tilde{f}|^p = \int_{\mathbb{R}^n} |f_k - \chi_E - \tilde{f}|^p \to 0 \quad \text{as } k \to \infty.$$ 

This proves that $\mathcal{D}'$ is dense in $L^p(E)$.
Given $h \in \mathbb{R}^n$. Let $\tau_h f(x) = f(x + h)$ be the translation operator. Similar to the case $L^1(\mathbb{R}^n)$, we have continuity of variable translations with respect to $\| \cdot \|_p$.

**Theorem 3.5.** If $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^n)$, then
\[
\lim_{h \to \infty} \| \tau_h f - f \|_p = 0.
\]

**Proof.** Let $C_p$ be the collection of $L^p(\mathbb{R}^n)$ functions satisfying this property. It follows easily from the Minkowski inequality that it is a subspace of $L^p(\mathbb{R}^n)$.

Given $E \subset \mathbb{R}^n$ with $m(E) < \infty$. By the Lebesgue dominated convergence theorem,
\[
\int_{\mathbb{R}^n} |\chi_E(x + h) - \chi_E(x)|^p dx = m(E \setminus E_h) + m(E_h \setminus E) \to 0 \text{ as } h \to 0,
\]
where $E_h = E - h = \{e - h : e \in E\}$. This shows that $\chi_E \in C_p$, and as well as simple functions in $L^p(\mathbb{R}^n)$. Suppose $f \in L^p(\mathbb{R}^n)$ is nonnegative. Choose simple functions $f_k \geq 0$, $f_k \not\to f$. Then $f_k \in L^p(\mathbb{R}^n)$ and, by the monotone convergence theorem, $\|f_k \to f\|_p \to 0$ as $k \to \infty$. Therefore,
\[
\|\tau_h f - f\|_p \leq \|\tau_h f - \tau_h f_k\|_p + \|\tau_h f_k - f_k\|_p + \|f_k - f\|_p
\]
\[
\leq \|\tau_h f_k - f_k\|_p + 2\|f_k - f\|_p
\]
Let $h \to 0$, then let $k \to \infty$, we find that $\limsup_{h \to \infty} \|\tau_h f - f\|_p = 0$. This proves that $f \in C_p$. This actually implies $C_p = L^p(\mathbb{R}^n)$ since any $f \in L^p(\mathbb{R}^n)$ is the difference of two nonnegative measurable functions in $L^p(\mathbb{R}^n)$.

**Exercises.**

3.1. Given any $f \in L^\infty(E)$. Show that $m(\{f > \text{ess sup}_E f\}) = 0$.

3.2. Use the generalized Young’s inequality in Exercise 2.2 to formulate a generalization of Hölder’s inequality for $L^p(E)$.

3.3. Suppose $m(E) < \infty$. Show that $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$. How about if $m(E) = \infty$?

3.4. Let $f$ be a real-valued measurable function on $E$. Define the *essential infimum* on $E$ by
\[
\text{ess inf} f = \sup\{\alpha \in (-\infty, \infty) : m(\{f < \alpha\}) = 0\}.
\]
Show that, if $f \geq 0$, then $\text{ess inf}_E f = 1/\text{ess sup}_E(1/f)$.

3.5. Consider $L^p(E)$ with $0 < p < 1$. Verify that $\rho_p(f, g) = \int_E |f - g|^p$ is a metric on $L^p(E)$.

3.6. Consider $L^p(E)$ with $0 < p < 1$. Suppose $f_n$ converges to $f$ almost everywhere. Prove that each of the following conditions implies $\|f_n - f\|_p \to 0$ as $n \to \infty$.

(a) There exists some $g \in L^p(E)$ such that $|f_n| \leq g$ for any $n$.
(b) $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$.

3.7. For what kind of $f \in L^p(E)$ and $g \in L^q(E)$, $\frac{1}{p} + \frac{1}{q} = 1$, do we have equality for the Hölder inequality? For what kind of $f, g \in L^p(E)$ do we have equality for the Minkowski inequality?

3.8. Consider $1 < p < \infty$. Give a proof for the Minkowski inequality using convexity of $x^p$, and without using Hölder’s inequality.

3.9. Show that $L^\infty(E)$ is not separable whenever $m(E) > 0$. 