CHAPTER 8

Introduction to Banach Spaces

1. Uniform and Absolute Convergence

As a preparation we begin by reviewing some familiar properties of Cauchy sequences and uniform limits in the setting of metric spaces.

**Definition 1.1.** A metric space is a pair \((X, \rho)\), where \(X\) is a set and \(\rho\) is a real-valued function on \(X \times X\) which satisfies that, for any \(x, y, z \in X\),

(a) \(\rho(x, y) \geq 0\) and \(\rho(x, y) = 0\) if and only if \(x = y\),
(b) \(\rho(x, y) = \rho(y, x)\),
(c) \(\rho(x, z) \leq \rho(x, y) + \rho(y, z)\). (Triangle inequality)

The function \(\rho\) is called the metric on \(X\).

Any metric space has a natural topology induced from its metric. A subset \(U\) of \(X\) is said to be open if for any \(x \in U\) there exists some \(r > 0\) such that \(B_r(x) \subset U\). Here \(B_r(x) = \{y \in X : \rho(x, y) < r\}\) is the open ball of radius \(r\) centered at \(x\). It is an easy exercise to show that open balls are indeed open and the collection of open sets is indeed a topology, called the metric topology.

On the contrary, there are topological spaces whose topology can be defined by some metric. In this case we say the topology is metrizable.

**Definition 1.2.** A sequence \(\{x_n\}\) in a metric space \((X, \rho)\) is said to be a Cauchy Sequence if

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \rho(x_n, x_m) < \varepsilon \text{ whenever } n, m \geq N.
\]

The metric space \((X, \rho)\) is said to be complete if every Cauchy sequence is convergent.

**Definition 1.3.** Let \((X, \rho)\) be a metric space. For any nonempty set \(A \subset X\), the diameter of the set \(A\) is defined by

\[
diam(A) = \sup\{\rho(x, y) : x, y \in A\}.
\]

The set \(A\) is said to be bounded if its diameter is finite. Otherwise, we say it is unbounded.

Let \(S\) be a nonempty set. We say a function \(f : S \to X\) is bounded if its image \(f(S)\) is a bounded set. Equivalently, it is bounded if for any \(x \in X\), there exists \(M > 0\) such that \(\rho(f(s), x) \leq M\) for any \(s \in S\). We say \(f\) is unbounded if it is not bounded.

**Definition 1.4.** Given a sequence \(\{f_n\}\) of functions from \(S\) to \(X\). We say \(\{f_n\}\) converges pointwise to the function \(f : S \to X\) if

\[
\forall s \in S, \forall \varepsilon > 0, \exists N_s \in \mathbb{N} \text{ such that } \rho(f_n(s), f(s)) < \varepsilon, \forall n \geq N_s.
\]

In this case, the function \(f\) is called the pointwise limit.

We say \(\{f_n\}\) converges uniformly to a function \(f : S \to X\) if the above \(N_s\) is independent of \(s\); that is,

\[
\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \rho(f_n(s), f(s)) < \varepsilon, \forall n \geq N, \forall s \in S.
\]
The function $f$ is called the uniform limit of $\{f_n\}$.

When $S$ is countably infinite, the function $f$ above is a sequence in $X$, and $\{f_n\}$ is a sequence of sequences in $X$; or in other words, $\{f_n(m)\}$ is a double sequence in $X$.

The next two theorems highlight some important features of Cauchy sequences and uniform convergence.

**Theorem 1.1.** (Cauchy Sequences) Consider sequences in a metric space $(X, \rho)$.

(a) Any convergent sequence is a Cauchy sequence.

(b) Any Cauchy sequence is bounded.

(c) If a subsequence of a Cauchy sequence converges, then the Cauchy sequence converges to the same limit.

**Proof.** (a) Suppose $\{x_n\}$ converges to $x$. Given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\rho(x_n, x) < \varepsilon/2$ for any $n \geq N$. The sequence $\{x_n\}$ is Cauchy because

$$\rho(x_n, x_m) < \rho(x_n, x) + \rho(x, x_m) < \varepsilon \quad \text{for any } n, m \geq N.$$

(b) Let $\{x_n\}$ be a Cauchy sequence. Choose $N \in \mathbb{N}$ such that $\rho(x_n, x_m) < 1$ for all $n, m \geq N$. Then for any $x \in X$,

$$\rho(x_n, x) \leq \rho(x_n, x_N) + \rho(x_N, x) < \max\{\rho(x_1, x_N), \rho(x_2, x_N), \ldots, \rho(x_{N-1}, x_N), 1\} + \rho(x_N, x),$$

where the last equation is a finite bound independent of $n$.

(c) Let $\{x_n\}$ be a Cauchy sequence with a subsequence $\{x_{n_k}\}$ converging to $x$. Given $\varepsilon > 0$, choose $K, N \in \mathbb{N}$ such that

$$\rho(x_{n_k}, x) < \frac{\varepsilon}{2} \quad \text{for any } k \geq K,$$

$$\rho(x_n, x_N) < \frac{\varepsilon}{2} \quad \text{for any } n, m \geq N.$$

Taking $n_k$ such that $k \geq K$ and $n_k \geq N$, then

$$\rho(x_n, x) \leq \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x) < \varepsilon \quad \text{for any } n \geq N.$$

This shows that $\{x_n\}$ converges to $x$. □

**Theorem 1.2.** (Uniform Convergence) Given a sequence of functions $\{f_n\}$ from a nonempty set $S$ to a metric space $(X, \rho)$. Suppose $\{f_n\}$ converges uniformly to a function $f : S \to X$.

(a) If each $f_n$ is bounded, then so is $f$.

(b) Assume $S$ is a topological space, $E \subset S$. If each $f_n$ is continuous on $E$, then so is $f$.

**Proof.** (a) Choose $N \in \mathbb{N}$ such that $\rho(f(s), f_n(s)) < 1$ for any $n \geq N$ and $s \in S$. Given $x \in X$, choose $M > 0$ such that $\rho(f_N(s), x) < M$ for any $s \in S$. Then $f$ is bounded since

$$\rho(f(s), x) \leq \rho(f(s), f_N(s)) + \rho(f_N(s), x) \leq 1 + M \quad \text{for any } t \in S.$$

(b) Given $\varepsilon > 0$, $e \in E$. Choose $N \in \mathbb{N}$ such that $\rho(f(s), f_n(s)) < \varepsilon/3$ for any $n \geq N$ and $s \in S$. For this particular $N$, $f_N$ is continuous at $e$, and so there is a neighborhood $U$ of $e$ such that $\rho(f_N(e), f_N(u)) < \varepsilon/3$ whenever $u \in U$. Then

$$\rho(f(e), f(u)) \leq \rho(f(e), f_N(e)) + \rho(f_N(e), f_N(u)) + \rho(f_N(u), f(u)) < \varepsilon \quad \forall u \in U.$$

Therefore $f$ is continuous at $e$, and is continuous on $E$ since $e \in E$ is arbitrary. □
Definition 1.5. A vector space $V$ over field $\mathbb{F}$ is called a normed vector space (or normed space) if there is a real-valued function $\| \cdot \|$ on $V$, called the norm, such that for any $x, y \in V$ and any $\alpha \in \mathbb{F}$,

(a) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
(b) $\|\alpha x\| = |\alpha| \|x\|$.
(c) $\|x + y\| \leq \|x\| + \|y\|$. (Triangle inequality)

A norm $\| \cdot \|$ of $V$ defines a metric $\rho$ on $V$ via $\rho(x, y) = \|x - y\|$. All concepts from metric and topological spaces are applicable to normed spaces.

There are multiple ways of choosing norms once a norm is selected. A trivial one is to multiply the original norm by a positive constant. Concepts like neighborhood, convergence, and completeness are independent of the choice of these two norms, and so we shall consider them equivalent norms. A more precise characterization of equivalent norms is as follows.

Definition 1.6. Let $V$ be a vector space with two norms $\| \cdot \|$, $\| \cdot \|'$. We say these two norms are equivalent if there exists some constant $c > 0$ such that

$$\frac{1}{c} \|x\|' \leq \|x\| \leq c \|x\|'$$

for any $x \in V$.

Example 1.1. The Euclidean space $\mathbb{F}^n$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, with the standard norm $\| \cdot \|$ defined by

$$\|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2}$$

is a normed space. Now consider two other norms of $\mathbb{F}^n$ defined by

$$\|x\|_\infty = \max\{|x_1|, \cdots, |x_n|\}, \quad \text{called the sup norm;}$$
$$\|x\|_1 = |x_1| + \cdots + |x_n|, \quad \text{called the 1-norm.}$$

Verifications for axioms of norms are completely straightforward.

In the case of sup norm, “balls” in $\mathbb{R}^n$ are actually cubes in $\mathbb{R}^n$ with faces parallel to coordinate axes. In the case of 1-norm, “balls” in $\mathbb{R}^n$ are cubes in $\mathbb{R}^n$ with vertices on coordinate axes. These norms are equivalent since

$$\|x\|_\infty \leq \|x\| \leq \|x\|_1 \leq n\|x\|_\infty.$$ 

Definition 1.7. A complete normed vector space is called a Banach space.

Example 1.2. Consider the Euclidean space $\mathbb{F}^n$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, with the standard norm $\| \cdot \|$. The normed space $(\mathbb{R}^n, \| \cdot \|)$ is complete since every Cauchy sequence is bounded and every bounded sequence has a convergent subsequence with limit in $\mathbb{R}^n$ (the Bolzano-Weierstrass theorem). The spaces $(\mathbb{R}^n, \| \cdot \|_1)$ and $(\mathbb{R}^n, \| \cdot \|_\infty)$ are also Banach spaces since these norms are equivalent.

Example 1.3. Given a nonempty set $X$ and a normed space $(Y, \| \cdot \|)$ over field $\mathbb{F}$. The space of functions from $X$ to $Y$ form a vector space over $\mathbb{F}$, where addition and scalar multiplication are defined in a trivial manner: Given two functions $f, g$, and two scalars $\alpha, \beta \in \mathbb{F}$, define $\alpha f + \beta g$ by

$$(\alpha f + \beta g)(x) = \alpha f(s) + \beta g(s), \quad x \in X.$$

Let $b(X, Y)$ be the subspace consisting of bounded functions from $X$ to $Y$. Define a real-valued function $\| \cdot \|_\infty$ on $b(X, Y)$ by

$$\|f\|_\infty = \sup_{x \in X} \|f(x)\|.$$
It is clearly a norm on $b(X,Y)$, also called the *sup norm*. Convergence with respect to the sup norm is clearly the same as uniform convergence.

If $(Y,\|\cdot\|)$ is a Banach space, then any Cauchy sequence $\{f_n\}$ in $b(X,Y)$ converges pointwise to some function $f : X \to Y$, since $\{f_n(x)\}$ is a Cauchy sequence in $Y$ for any fixed $x \in X$. In fact, the convergence $f_n \to f$ is uniform. To see this, let $\varepsilon > 0$ be arbitrary. Choose $N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon/2$ whenever $n, m \geq N$. For any $x \in X$, there exists some $m_x \geq N$ such that $\|f_{m_x}(x) - f(x)\| < \varepsilon/2$. Then for any $n \geq N$,

$$\|f_n(x) - f(x)\| \leq \|f_n(x) - f_{m_x}(x)\| + \|f_{m_x}(x) - f(x)\| < \varepsilon.$$

This proves that the convergence $f_n \to f$ is uniform. By Theorem 1.2(a), $f \in b(X,Y)$, and so the space $b(X,Y)$ with the sup norm is a Banach space.

**Example 1.4.** Let $(X,T)$ be a topological space and let $(Y,\|\cdot\|)$ be a Banach space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Denote by $C(X,Y)$ the space of continuous functions from $X$ to $Y$. Let $C_b(X,Y) = C(X,Y) \cap b(X,Y)$, the space of bounded continuous functions from $X$ to $Y$. Given any sequence $\{f_n\}$ in $C_b(X,Y)$ which converges uniformly to $f \in b(X,Y)$. By Theorem 1.2(b), $f \in C_b(X,Y)$. This shows that $C_b(X,Y)$ is a closed subspace of $b(X,Y)$, and is therefore a Banach space (see Exercise 1.1).

**Definition 1.8.** A series $\sum_{k=1}^{\infty} a_k$ in a normed space $X$ is said to be convergent (or summable) if its partial sum $\sum_{k=1}^{n} a_k$ converges to some $s \in X$ as $n \to \infty$. We say $\sum_{k=1}^{\infty} a_k$ is absolutely convergent (or absolutely summable) if $\sum_{k=1}^{\infty} |a_k| < \infty$.

In the following we prove some useful criteria for completeness and uniform convergence of series.

**Theorem 1.3.** A normed space $X$ is complete if and only if every absolutely convergent series is convergent.

**Proof.** Suppose $X$ is complete, $\sum_{k=1}^{\infty} a_k$ is absolutely convergent. We need to show the convergence of $s_n = \sum_{k=1}^{n} a_k$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} |a_k| < \varepsilon$, then $\|s_n - s_m\| < \varepsilon$ whenever $n, m \geq N$. Thus $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and so it converges.

Conversely, suppose every absolutely convergent series in $X$ converges. Let $\{s_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $X$. By Theorem 1.1 it suffices to show that $\{s_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{s_{n_k}\}_{k=1}^{\infty}$. Now choose $n_k$ such that

$$n_k < n_{k+1}, \quad \|s_{n_k} - s_{n_{k+1}}\| < \frac{1}{2^k} \quad \text{for any } k \in \mathbb{N}.$$

Then the series $s_{n_1} + \sum_{k=1}^{\infty} (s_{n_{k+1}} - s_{n_k})$ converges absolutely, so that it converges to some $s \in X$. This implies that $s_{n_k} = s_{n_1} + \sum_{j=1}^{k-1} (s_{n_{j+1}} - s_{n_j})$ converges to $s$ as $k \to \infty$, completing the proof.

**Corollary 1.4.** (Weierstrass M-test)

Let $b(X,Y)$ be the space bounded functions from a nonempty set $X$ to a Banach space $(Y,\|\cdot\|)$. Given a sequence of functions $\{f_n\}_{n=1}^{\infty}$ in $b(X,Y)$. If $\|f_n\|_\infty \leq M_n$ for any $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

**Proof.** The assumption says that $\sum_{n=1}^{\infty} f_n$ is absolutely convergent. By Theorem 1.3 (and Example 1.3), the series converges in $b(X,Y)$, implying that the convergence is uniform. □
Exercises.

1.1. Show that a subset of a complete metric space is complete if and only if it is closed.

1.2. Let \( \Sigma_2 = \{0, 1\}^\mathbb{N} \), the space of infinite sequences of \{0, 1\}; that is,

\[ \Sigma_2 = \{(a_1, a_2, \ldots) : a_k = 0 \text{ or } 1 \text{ for each } k\} \]

Given \( \lambda > 1 \), \( a, b \in \Sigma_2 \), let

\[ \rho_\lambda(a, b) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{\lambda^k}. \]

Show that \((\Sigma_2, \rho_\lambda)\) is a complete metric space.

1.3. Consider the space of real sequences \( s \). Let

\[ \rho(a, b) = \sum_{k=1}^{\infty} \frac{|a_k - b_k|}{2^k(1 + |a_k - b_k|)}. \]

Show that \((s, \rho)\) is a complete metric space.

1.4. Consider the space \( BV[a, b] \) of functions on \([a, b]\) with bounded variations. For any \( f \in BV[a, b] \), let \( \|f\| = |f(a)| + V^b_a(f) \). Show that \((BV[a, b], \| \cdot \|)\) is a Banach space. Is it separable?

2. The \( \ell^p \) Space

**Definition 2.1.** Let \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \). Given \( 0 < p < \infty \), define

\[ \ell^p = \{ a = (a_1, a_2, \cdots) : a_k \in \mathbb{F} \text{ for any } k, \sum_k |a_k|^p < \infty \}, \quad \|a\|_p = \left( \sum_k |a_k|^p \right)^{\frac{1}{p}}; \]

\[ \ell^\infty = \{ a = (a_1, a_2, \cdots) : a_k \in \mathbb{F} \text{ for any } k, \sup_k |a_k| < \infty \}, \quad \|a\|_\infty = \sup_k |a_k|. \]

The space \( \ell^\infty \) consists of bounded sequences in \( \mathbb{F} \). Addition and multiplication of sequences are defined componentwise:

\[ (a_1, a_2, \cdots) + (b_1, b_2, \cdots) = (a_1 + b_1, a_2 + b_2, \cdots) \]
\[ (a_1, a_2, \cdots) \cdot (b_1, b_2, \cdots) = (a_1b_1, a_2b_2, \cdots). \]

Clearly \( \ell^p \) with any \( 0 < p \leq \infty \) is a vector space, since \( \|\alpha a\|_p = |\alpha|\|a\|_p \) for any \( \alpha \in \mathbb{F} \) and

\[ a, b \in \ell^p \Rightarrow \sum_k |a_k + b_k|^p \leq \sum_k (2\max\{|a_k|, |b_k|\})^p \leq 2^p \sum_k (|a_k|^p + |b_k|^p), \]
\[ a, b \in \ell^\infty \Rightarrow \sup_k |a_k + b_k| \leq \sup_k |a_k| + \sup_k |b_k|. \]

When \( 1 \leq p \leq \infty \), the function \( \| \cdot \|_p \) is a norm on \( \ell^p \). This follows from Theorem 2.1 below.

**Theorem 2.1.** Given \( 1 \leq p, q \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( a, b \) be sequences of complex numbers.

(a) (Young’s inequality) If \( u, v \geq 0, 1 < p, q < \infty \), then

\[ uv \leq \frac{u^p}{p} + \frac{v^q}{q}. \]
(b) **Hölder’s Inequality for \( \ell^p \)** If \( a \in \ell^p, b \in \ell^q \), then \( ab \in \ell^1 \) and

\[
\|ab\|_1 \leq \|a\|_p \|b\|_q.
\]

**Proof.** It would be convenient to write \( q = \frac{p}{p-1} \) (\( q = \infty \) if \( p = 1, q = 1 \) if \( p = \infty \)). The curve \( y = x^{p-1} \) can be alternatively written \( x = y^{q-1} \). Part (a) follows easily by observing

\[
U = \int_0^u x^{p-1}dx = \frac{u^p}{p}, \quad V = \int_0^u y^{q-1}dy = \frac{v^q}{q}, \quad U + V \geq uv.
\]

For part (b), the cases \( p = 1, q = \infty \) and \( p = \infty, q = 1 \) are obvious. Consider \( 1 < p, q < \infty \). The cases \( \|a\|_p = 0 \) or \( \|b\|_q = 0 \) are also obvious, so we assume \( 0 < \|a\|_p, \|b\|_q < \infty \).

Let \( A = \frac{a}{\|a\|_p}, B = \frac{b}{\|b\|_q} \). By Young’s inequality,

\[
\sum_k |A_k B_k| \leq \sum_k \left( \frac{|A_k|^p}{p} + \frac{|B_k|^q}{q} \right) = \frac{\|A\|^p_p}{p} + \frac{\|B\|^q_q}{q} = \frac{1}{p} + \frac{1}{q} = 1,
\]

\[
\|ab\|_1 = \sum_k |a_k b_k| = \|a\|_p \|b\|_q \sum_k |A_k B_k| \leq \|a\|_p \|b\|_q.
\]

The cases \( p = 1 \) and \( p = \infty \) for (c) are obvious. For \( 1 < p < \infty \), (c) follows easily from (b):

\[
\|a + b\|_p^p = \sum_k |a_k + b_k|^p \leq \sum_k |a_k|^p |b_k|^{p-1} + \sum_k |a_k + b_k|^{p-1} |b_k| \\
\leq \left( \sum_k |a_k|^p \right)^{p-1} \left( \sum_k |b_k|^p \right)^{1/p} + \left( \sum_k |a_k + b_k|^p \right)^{p-1} \left( \sum_k |b_k|^p \right)^{1/p} \\
= \|a\|_p \|b\|_p \|a + b\|_p^p - \|ab\|_1 \|a\|_p \|b\|_p.
\]

\[\Box\]

**Theorem 2.2.** For any \( 1 \leq p \leq \infty \), \((\ell^p, \| \cdot \|_p)\) is a Banach space.

**Proof.** Completeness of \( \ell^\infty \) is a special case of Example 1.3. Consider \( 1 \leq p < \infty \). Let \( \{a^{(n)}\}_{i=1}^\infty \) be a Cauchy sequence in \( \ell^p \). For each \( k \), \( \{a^{(n)}_k\}_{n=1}^\infty \) is a Cauchy sequence of real numbers since

\[
|a^{(n)}_k - a^{(m)}_k| \leq \left( \sum_{j=1}^\infty |a^{(n)}_j - a^{(m)}_j|^p \right)^{1/p} = \|a^{(n)} - a^{(m)}\|_p.
\]

Then there is a sequence \( a = (a_1, a_2, \cdots) \) such that, for each \( k \),

\[
a^{(n)}_k \to a_k \in \mathbb{R} \quad \text{as } n \to \infty.
\]
Given $\varepsilon > 0$, $M \in \mathbb{N}$, there exists some $N \in \mathbb{N}$ such that
\[
\left( \sum_{k=1}^{M} \left| a_k^{(n)} - a_k^{(m)} \right|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{\infty} \left| a_k^{(n)} - a_k^{(m)} \right|^p \right)^{\frac{1}{p}} < \varepsilon \quad \forall \ n > m \geq N.
\]
Let $m \to \infty$, then let $M \to \infty$, we find
\[
\|a^{(n)} - a\|_p = \left( \sum_{k=1}^{\infty} \left| a_k^{(n)} - a_k \right|^p \right)^{\frac{1}{p}} \leq \varepsilon \quad \text{for any } n \geq N.
\]
Thus $\|a^{(n)} - a\|_p \to 0$ as $n \to \infty$, and so $\|a\|_p \leq \|a - a^{(n)}\|_p + \|a^{(n)}\|_p < \infty$, $a \in \ell^p$. This verifies completeness of $\ell^p$.

**Theorem 2.3.** The space $\ell^p$ is separable if $1 \leq p < \infty$, and the space $\ell^\infty$ is not separable.

**Proof.** The space $\ell^\infty$ is not separable because it has an uncountable subset $s = \{a = (a_1, a_2, \ldots) \in \ell^\infty : a_n = 0 \text{ or } 1 \forall \ n\}$ and $\|a - b\|_\infty = 1$ for any $a \neq b \in s$.

Consider $1 \leq p < \infty$. Let $\mathcal{D}$ be the set of finite sequences with rational coordinates. Clearly $\mathcal{D}$ is countable. Given $a \in \ell^p$ and any $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} |a_k|^p < \varepsilon^p/2$. Now choose $b_1, \ldots, b_N \in \mathbb{Q}$ such that $\sum_{k=1}^{N} |a_k - b_k|^p < \varepsilon^p/2$. Let $b = (b_1, \ldots, b_N, 0, 0, \ldots) \in \mathcal{D}$. Then
\[
\|a - b\|^p_p = \sum_{k=1}^{N} |a_k - b_k|^p = \sum_{k=1}^{N} |a_k - b_k|^p + \sum_{k=N+1}^{\infty} |a_k|^p < \varepsilon^p.
\]
Thus $\|a - b\|_p < \varepsilon$. This shows that $\mathcal{D}$ is dense since $\varepsilon > 0$ is arbitrary.

**Exercises.**

2.1. Consider the $\ell^p$ space with $0 < p < 1$. Verify that $\rho_p(a, b) = \sum_{k=1}^{\infty} |a_k - b_k|^p$ is a metric on $\ell^p$. Prove that $(\ell^p, \rho_p)$ is a complete separable metric space.

2.2. Prove the following generalization of Young’s inequality: Given $1 < p_1, \ldots, p_n < \infty$ with $\sum_{k=1}^{n} \frac{1}{p_k} = 1$. If $u_1, \ldots, u_n \geq 0$, then
\[
u_1 \cdots u_n \leq \frac{u_1^{p_1}}{p_1} + \cdots + \frac{u_n^{p_n}}{p_n}.
\]
Use it to formulate a generalization of Hölder’s inequality for $\ell^p$.

2.3. Consider sequences of real numbers. Show that the space $c_0$ of sequences converging to zero with sup norm is a Banach space, and for any $1 \leq p < q \leq \infty$, $a \in \ell^p$,
\[
\ell^p \subsetneq \ell^q \subsetneq c_0, \quad \|a\|_\infty \leq \|a\|_q \leq \|a\|_p.
\]
Are these norms on $\ell^p$ equivalent?

2.4. Explain why the set $s$ in the proof of Theorem 2.3 is uncountable.