Show detailed argument to each problem.

1. (10 points) Suppose \( E \) is a Lebesgue measurable subset of \( \mathbb{R} \) with \( |E| < \infty \). Prove that
\[
|E| = \sup \{|K| : K \subset E \text{ and } K \text{ is compact}\}.
\] (0.1)

**solution:**
Let \( E_n = E^T \cap [-n, n] \), \( n \in \mathbb{N} \). We have \( E_n \not\subset E \) and so \( |E - E_n| \to 0 \) as \( n \to \infty \) (note that \( |E| < \infty \)). For any \( \varepsilon > 0 \) one can find \( m \in \mathbb{N} \) so that \( |E - E_n| < \varepsilon / 2 \) for all \( n \geq m \). For \( E_m \) one can find a closed subset \( F \subset E_m \) such that \( |E_m - F| < \varepsilon / 2 \). Hence \( |E - F| \leq |E - E_m| + |E_m - F| < \varepsilon \). In particular \( F \) is compact since \( E_m \) is bounded. We also have
\[
|E| \leq |E - F| + |F| < \varepsilon + |F|
\]
where \( F \subset E \) and \( F \) is compact. (0.1) is proved.

2. (10 points) Assume \( E (t) \) is a continuously differentiable increasing function on \([0, \infty)\) such that \( 0 \leq E (t) \leq C \) for all \( t \in [0, \infty) \), where \( C \) is some positive constant. Show that for any \( \varepsilon > 0 \), we have
\[
\left( \int_{t \in [0, \infty)} : E' (t) > \varepsilon \right) \leq \frac{C}{\varepsilon}.
\]

**solution:**
By Chebyshev inequality we have
\[
\left( \int_{t \in (0, \infty)} : E' (t) > \varepsilon \right) \leq \frac{1}{\varepsilon} \int_0^\infty E' (t) dt \leq \frac{1}{\varepsilon} \lim_{t \to \infty} E (t) - E (0) \leq \frac{C}{\varepsilon}
\]
since \( 0 \leq E (t) \leq C \) for all \( t \in [0, \infty) \).

3. (10 points) Let \( f_n : E \to \mathbb{R} \) be a sequence of measurable functions defined on a measurable set \( E \subset \mathbb{R}^n \). Let
\[
A = \{ x \in E : \lim_{n \to \infty} f_n (x) \text{ exists } \}.
\]
Is \( A \) a measurable set or not? Give your reasons.

**solution:**
Let \( F^* (x) = \limsup_{n \to \infty} f_n (x) \) and \( F_* (x) = \liminf_{n \to \infty} f_n (x) \). We know that both functions are measurable on \( E \). Hence the sets
\[
S_1 := \{ x \in E : F^* (x) = \infty \text{ and } F_* (x) = \infty \}
\]
\[
S_2 := \{ x \in E : F^* (x) = -\infty \text{ and } F_* (x) = -\infty \}
\]
are all measurable. Let \( S = S_1 \cup S_2 \). Now \( E - S \) is also measurable, and \( F^* - F_* \) is a measurable function on \( E - S \). By the relation
\[
A = \{ x \in E - S : F^* (x) - F_* (x) = 0 \}
\]
we know that \( A \) is a measurable set.

4. (10 points) Give an example of a sequence of measurable functions \( \{ f_k \} \) defined on a measurable set \( E \subset \mathbb{R}^n \) such that the following strict inequalities hold:
\[
\liminf_{k \to \infty} f_k dx < \liminf_{k \to \infty} f_k dx < \limsup_{k \to \infty} f_k dx < \limsup_{k \to \infty} f_k dx.
\]

**solution:**
On the interval $[0, 1]$, let

$$
f_k(x) = \begin{cases} 
1, & x \in \left[0, \frac{1}{2}\right] \\
0, & x \in \left(\frac{1}{2}, 1\right]
\end{cases}, \quad k = 1, 3, 5, 7, \ldots
$$

and

$$
f_k(x) = \begin{cases} 
0, & x \in \left[0, \frac{1}{2}\right] \\
2, & x \in \left(\frac{1}{2}, 1\right]
\end{cases}, \quad k = 2, 4, 6, 8, \ldots
$$

then

$$
0 = \liminf_{k \to \infty} \int_E f_k \, dx < \liminf_{k \to \infty} \int_E f \, dx = \frac{1}{2},
$$

and

$$
1 = \limsup_{k \to \infty} \int_E f_k \, dx < \limsup_{k \to \infty} \int_E f \, dx = \frac{3}{2}.
$$

5. (15 points) Assume $f \in L[a, b]$ and let $h(x) = \frac{1}{a} \int_a^x f, \ x \in [a, b]$. Is the function $h(x)$ a measurable function on $[a, b]$? Give your reasons.

**solution:**

By looking at $f^+$ and $f^-$, without loss of generality, we may assume that $f \geq 0$ on $[a, b]$. Now there exists a sequence of simple functions $0 \leq f_k \not\to f$ a.e. on $E$, where $f_k \in L[a, b]$ also for all $k$. Monotone Convergence Theorem implies

$$
\int_a^x f_k \, dx \to \int_a^x f \, dx \quad \text{as} \quad k \to \infty
$$

for all $x \in [a, b]$. As $f_k$ is a bounded function on $[a, b]$, we know that $\frac{1}{a} f_k$ is a continuous function of $x \in [a, b]$. Hence $h(x)$ is a measurable function on $[a, b]$.

Another solution:

We have $h(x) = \frac{1}{a} \int_a^x f^+ - \frac{1}{a} \int_a^x f^-$ and both $\frac{1}{a} \int_a^x f^+$ and $\frac{1}{a} \int_a^x f^-$ are increasing functions on $[a, b]$. We know that an increasing function on $[a, b]$ is continuous a.e. on $[a, b]$. Hence $h(x)$ is a measurable function on $[a, b]$.

6. (15 points) Suppose $E \subseteq \mathbb{R}$ is measurable with $|E| = \lambda > 0$, where $\lambda$ is a finite number. Show that for any $t$ with $0 < t < \lambda$, there exists a subset $A$ of $E$ such that $A$ is measurable and $|A| = t$. That is, the Lebesgue measure $|\cdot|$ on $\mathbb{R}$ satisfies the Intermediate Value Theorem.

**solution:**

Define $f(x) = \int_{(-\infty, x)}^T E \setminus x \in \mathbb{R}$. Then for $x < y$ we have $f(x) \leq f(y)$ and

$$
f(y) - f(x) = \int_{(-\infty, y)}^T E \setminus (-\infty, x) \setminus E^c \setminus (-\infty, y) \setminus E^c \setminus \int_{(-\infty, x)}^T E \setminus y - x.
$$

This means that $f(x)$ is a continuous function on $\mathbb{R}$ with $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = \lambda$. For any $0 < t < \lambda$, by the mean value theorem of continuous functions, we have $f(s) = t$ for some $s \in \mathbb{R}$. Hence the set $A = (-\infty, s)^T E$ satisfies $|A| = t$.

7. (15 points) Let $y = Tx$ be a nonsingular linear transformation of $\mathbb{R}^n$. If $\int_E f(y) \, dy$ exists, show that we have the following change of variables formula:

$$
\int_E f(y) \, dy = |\det T| \int_{T^{-1}E} f(Tx) \, dx.
$$

(0.2)
solution:

**Case1:** If \( f(x) = \chi_{E_1} \), \( E_1 \subset E \), then LHS of (0.2) is \( |E_1| \), and the RHS is given by

\[
|\det T|_{T^{-1}E} \chi_{E_1}(Tx) dx = |\det T| \cdot T^{-1}E_1.
\]

We see that (0.2) holds by Theorem 3.35.

**Case2:** Assume \( f \geq 0 \). Then there exists a sequence of simple functions \( 0 \leq s_n \not\equiv f \) on \( E \) where

\[ s_n = a_1\chi_{E_1} + \cdots + a_{k(n)}\chi_{E_{k(n)}}, \quad k(n) \text{ depends on } n. \]

Now by Case1

\[
s_n(y) dy = a_1 Z_{E_1} \chi_{E_1}(y) dy + \cdots + a_{k(n)} Z_{E_{k(n)}} \chi_{E_{k(n)}}(y) dy
\]

\[
= a_1 |\det T|_{T^{-1}E} \chi_{E_1}(Tx) dx + \cdots + a_{k(n)} |\det T|_{T^{-1}E} \chi_{E_{k(n)}}(Tx) dx
\]

\[
= |\det T|_{T^{-1}E} s_n(Tx) dx
\]

and by the **Monotone Convergence Theorem** we obtain

\[
\lim_{n \to \infty} s_n(y) dy = f(y) dy, \quad \lim_{n \to \infty} |\det T|_{T^{-1}E} s_n(Tx) dx = |\det T|_{T^{-1}E} f(Tx) dx.
\]

The conclusion follows.

For general \( f \), use

\[
f = f^+ - f^- = |\det T|_{T^{-1}E} f^+(Tx) dx - |\det T|_{T^{-1}E} f^-(Tx) dx
\]

\[
= |\det T|_{T^{-1}E} f(Tx) dx.
\]

**Remark 1** (be careful) \( E \)

\[
|f_k - f|^p \to 0 \text{ as } k \to \infty \text{ does not, in general, imply that } |f_k - f|^p \to 0 \text{ a.e. on } E \text{ as } k \to \infty. \]

Also for \( p > 0 \) the inequality

\[
|f|^p \leq |f_k - f|^p + |f_k|^p
\]

is wrong in general. It holds only for \( 0 < p \leq 1. \)