1. (10 points) Do Exercise 6 in p. 85.

**Solution:**

In this problem we assume \( \frac{\partial}{\partial x} f(x,y) \) exists on \( I = [0,1] \times [0,1] \). We also know that it is a bounded function on \( I \). Let

\[
F_n(x,y) = \frac{f \left( x + \frac{1}{n}, y \right) - f(x,y)}{\frac{1}{n}}, \quad n = 1, 2, 3, \ldots.
\]

We see that for each fixed \( x \), \( F_n(x,y) \) is a sequence of bounded (use mean value theorem to see this) measurable functions of \( y \). By

\[
\frac{\partial}{\partial x} f(x,y) = \lim_{n \to \infty} F_n(x,y)
\]

we know that for each fixed \( x \), \( \frac{\partial}{\partial x} f(x,y) \) is a measurable function of \( y \). Now by the Bounded Convergence Theorem, we obtain

\[
\frac{d}{dx} \int_0^1 f(x,y) \, dy = \lim_{n \to \infty} \int_0^1 F_n(x,y) \, dy = \int_0^1 \frac{\partial}{\partial x} f(x,y) \, dy.
\]

\[ \square \]

2. (10 points) Do Exercise 9 in p. 85.

**Solution:**

For any \( \varepsilon > 0 \) we have

\[
|\{(f_k - f)^p > \varepsilon\}| \leq \frac{1}{\varepsilon} \int_E |f_k - f|^p \to 0 \quad \text{as} \quad k \to \infty
\]

due to the Tchebyshev inequality. Hence \( f_k \) converges to \( f \) in measure on \( E \).

\[ \square \]


**Solution:**

By Exercise 9 we know that \( f_k \to f \) in measure. In particular, there exists a subsequence \( f_{k_j} \) such that it converges to \( f \) a.e. on \( E \). Fatou's lemma implies

\[
\liminf_{j \to \infty} f_{k_j}^p = \frac{1}{E} \liminf_{j \to \infty} |f_{k_j}|^p \leq \frac{1}{E} |f|^p \leq \liminf_{j \to \infty} |f_{k_j}|^p \leq M.
\]

\[ \square \]


**Solution:**

**Case 1:** If \( f(x) = \chi_{E_1} \), \( E_1 \subset E \), then LHS of the identity is \( |E_1| \), and the RHS of the identity is given by

\[
\int_{T^{-1}E} \chi_{E_1}(Tx) \, dx = |\det T| \cdot T^{-1}E_1.
\]
We see that the identity holds by Theorem 3.35.

**Case 2:** Assume $f \geq 0$. Then there exists a sequence of simple functions $0 \leq s_n \nearrow f$ on $E$ where

$$s_n = a_1 \chi_{E_1} + \cdots + a_{k(n)} \chi_{E_{k(n)}}, \quad k(n) \text{ depends on } n.$$

Now by Case 1

$$Z_E s_n(y) dy = Z_E a_1 \chi_{E_1} (y) dy + \cdots + a_{k(n)} \chi_{E_{k(n)}} (y) dy$$

$$= a_1 |\det T| Z_{T^{-1}E} \chi_{E_1} (Tx) dx + \cdots + a_{k(n)} |\det T| Z_{T^{-1}E} \chi_{E_{k(n)}} (Tx) dx$$

$$= |\det T| Z_{T^{-1}E} a_1 \chi_{E_1} + \cdots + a_{k(n)} \chi_{E_{k(n)}} (Tx) dx$$

and by the Monotone Convergence Theorem we obtain

$$\lim_{n \to \infty} Z_E s_n(y) dy = f(y) dy, \quad \lim_{n \to \infty} |\det T| Z_{T^{-1}E} s_n(Tx) dx = |\det T| Z_{T^{-1}E} f(Tx) dx.$$

The conclusion follows.

For general $f$, use

$$f = f^+ - f^- = |\det T| Z_{T^{-1}E} f^+(Tx) dx - |\det T| Z_{T^{-1}E} f^-(Tx) dx$$

$$= |\det T| Z_{T^{-1}E} f(Tx) dx.$$