1. (10 points) Let \( f : E \to \mathbb{R} \) be a nonnegative measurable function such that \( \int_E f < \infty \). Show that for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any measurable subset \( E_1 \subseteq E \) with \( |E_1| < \delta \) we have \( \int_{E_1} f < \varepsilon \).

**Solution:** Let
\[
  f_k(x) = \begin{cases} f(x), & \text{if } f(x) < k \\ k, & \text{if } f(x) \geq k \end{cases}, \quad x \in E.
\]
Then \( 0 \leq f_k(x) \leq f(x) \) on \( E \) and by the Monotone Convergence Theorem we have
\[
  \lim_{k \to \infty} \int_E f_k(x) = \int_E f < \infty
\]
and so for any \( \varepsilon > 0 \) there exists \( N \) such that for any \( \varepsilon > 0 \) there exists \( N \) such that
\[
  \int_{E_1} f_N < \varepsilon / 2.
\]
Therefore for any \( E_1 \subseteq E \) with \( |E_1| < \delta := \frac{\varepsilon}{2} \) we would have
\[
  \int_{E_1} f \leq \int_{E_1} f_N < \varepsilon.
\]

2. (10 points) Do Exercise 3 in p. 85.

**Solution:** Since \( f_k \leq f \) a.e. on \( E \) (both are nonnegative), we have \( \int_E f_k \leq \int_E f \) for all \( k \). On the other hand, by Fatou’s lemma we get
\[
  \liminf_{k \to \infty} \int_E f_k = \int_E f \leq \liminf_{k \to \infty} \int_E f_k
\]
which implies
\[
  \int_E f \leq \limsup_{k \to \infty} \int_E f_k \leq \int_E f \leq \int_E f dx \leq \int_E f.
\]
Hence we have \( \lim_{k \to \infty} \int_E f_k = \int_E f \).

3. (10 points) Let \( f_k : E \to \mathbb{R} \) be a sequence of nonnegative measurable function satisfying \( \int_E f_k \to 0 \) as \( k \to \infty \). Show that \( f_k \to 0 \) in measure as \( k \to \infty \).

**Solution:** For any \( \varepsilon > 0 \) by Tchebyshev’s inequality we have
\[
  |\{ x \in E : |f_k - 0| > \varepsilon \}| = |\{ x \in E : f_k > \varepsilon \}| \leq \frac{1}{\varepsilon} \int_E f_k.
\]
Letting \( k \to \infty \), the conclusion follows.

**Remark 1** (be careful) \( \int_E f_k \to 0 \) as \( k \to \infty \) does not, in general, imply that \( f_k \to 0 \) a.e. on \( E \).

4. (10 points) Compute the limit
\[
  \lim_{n \to \infty} \int_0^3 1 - \frac{x}{n} e^{x/2} dx
\]
and justify your answer.

**Solution:** Let
\[
  f_n(x) = \begin{cases} 1 - \frac{x}{n} e^{x/2}, & \text{if } x \leq n \\ 0, & \text{if } x > n. \end{cases}
\]
One can check that \( f_n(x) \leq f(x) = e^{-x/2} \) on \( E = [0, \infty) \). By Monotone Convergence Theorem we have
\[
  \lim_{n \to \infty} \int_E f_n = \int_E f = \int_0^\infty e^{-x/2} dx = 2.
\]