1. (10 points) Do Exercise 2 in p. 61.

Solution: Assume \( f \) takes distinct values \( a_1, \ldots, a_N \) on disjoint sets \( E_1, \ldots, E_N \). We can express the function \( f(x) \) as

\[
f(x) = a_1 \chi_{E_1}(x) + \cdots + a_N \chi_{E_N}(x), \quad x \in E = \bigcup_{k=1}^{N} E_k.
\]

If each \( E_k \) is measurable, then the characteristic function \( \chi_{E_k}(x) : E \to \mathbb{R} \) is a measurable function on \( E \). Hence if \( E_1, \ldots, E_N \) are all measurable, so is \( f(x) \).

Conversely, assume \( f(x) \) is measurable. Then by definition we know that (we may assume \( a_1 < a_2 < \cdots < a_N \)) the set \( \{ f > a_{N-1} \} \) is measurable. Since \( \{ f > a_{N-1} \} = E_N \), the set \( E_N \) is measurable. Similarly by

\[
E_{N-1} = \{ f > a_{N-2} \} - E_N
\]

we know that \( E_{N-1} \) is measurable. Keep going to conclude that \( E_1, \ldots, E_N \) are all measurable.

2. (10 points) Do Exercise 3 in p. 61.

Solution: Let \( F(x) = (f(x), g(x)), \ x \in \mathbb{R}^n, \ F : \mathbb{R}^n \to \mathbb{R}^2. \)

(\( \Rightarrow \)) Assume \( F \) is measurable. For any open set \( G_x \subset \mathbb{R} \), let \( G = G_x \times \mathbb{R} \). Then \( G \subset \mathbb{R}^2 \) is open and by the identity

\[
F^{-1}(G) = f^{-1}(G_x)
\]

we know that \( f^{-1}(G_x) \) is measurable for any open set \( G_x \subset \mathbb{R} \). Hence \( f : \mathbb{R}^n \to \mathbb{R} \) is measurable. The same for \( g : \mathbb{R}^n \to \mathbb{R} \).

(\( \Leftarrow \)) Assume \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) are both measurable. Let \( G \subset \mathbb{R}^2 \) be an arbitrary open set. We can express \( G \) as a countable union of nonoverlapping closed intervals \( G = \bigcup_{k=1}^{\infty} I_k \), where \( I_k = [a_k,b_k] \times [c_k,d_k] \). Note that

\[
F^{-1}(I_k) = f^{-1}[a_k,b_k] \cap g^{-1}[c_k,d_k] \quad (\text{both sets are measurable})
\]

and so \( F^{-1}(I_k) \) is measurable. By

\[
F^{-1}(G) = \bigcup_{k=1}^{\infty} F^{-1}(I_k)
\]

the set \( F^{-1}(G) \) is measurable in \( \mathbb{R}^n \). Hence \( F \) is a measurable function.

3. (10 points) Do Exercise 4 in p. 61.

Solution: For any \( a \in \mathbb{R} \), we have

\[
\{ x \in \mathbb{R}^n : f(Tx) > a \} = \{ x \in \mathbb{R}^n : f(Tx) \in (a, \infty) \} \subset \bigcup_{\mathfrak{a}} x \in \mathbb{R}^n : T^{-1}f^{-1}(\{ a \}) = \bigcup_{\mathfrak{a}} x \in \mathbb{R}^n : T^{-1}f^{-1}(\{ \infty \}) \subset \mathbb{R}^n .
\]

As \( T^{-1} : \mathbb{R}^n \to \mathbb{R}^n \) is linear and Lipschitz continuous, \( \mathfrak{a} \) is measurable. Hence \( f(Tx) \) is a measurable function.

\[ \]
4. (20 points) Do Exercise 5 in p. 61.

**First solution:** We first prove the following

**Lemma 0.1** The Cantor-Lebesgue function \( f(x) : [0, 1] \to [0, 1] \) satisfies \( f(x_1) = f(x_2) \), where \( x_1 \) and \( x_2 \) are in \( C \), if and only if both \( x_1 \) and \( x_2 \) are endpoints of some interval removed.

**Proof.** The direction \((\Leftarrow)\) is trivial.

\((\Rightarrow)\) Assume at least one of \( x_1, x_2 \) is not endpoint, say \( x_2 \). On the interval \((x_1, x_2)\), \( x_1 < x_2 \), \( x_1, x_2 \in C \), there exists some \( p \notin C \). Let \( I = (y_1, y_2) \) be the maximal open interval containing \( p \) such that \( I \cap C = \emptyset \) (note that the complement of \( C \) is open). We now have \( y_1, y_2 \in C \) and \( y_2 < x_2 \) (otherwise if \( y_2 = x_2 \), then \( x_2 \) must be an endpoint, impossible). Similarly one can find an open interval \( J = (z_1, z_2) \) such that \( J \subset (y_2, x_2) \) with \( y_2 < z_1 \) (otherwise \( y_2 \) is an isolated point of \( C \), impossible). Now the open interval \( J \) is on the right hand side of the open interval \( I \) with a positive distance away. These two distinct intervals must be exactly equal to some removed intervals in the process of constructing the Cantor set. Hence \( f(I) < f(J) \), which gives \( f(x_1) < f(x_2) \).

**Corollary 0.2** If \( x \in C \) but \( x \) is not endpoint of some interval removed (say \( x = \frac{1}{3} \)), then there is no \( \tilde{x} \in C \), \( \tilde{x} \neq x \), such that \( f(\tilde{x}) = f(x) \).

**Corollary 0.3** Let \( \tilde{C} = C - \{ \text{all right endpoints of the removed intervals} \} \), where \( C \) is the Cantor set. Then \( f : \tilde{C} \to [0, 1] \) is 1-1 and onto, and is strictly increasing on \( \tilde{C} \). Here \( f \) is the Cantor-Lebesgue function.

By the above corollary, we have \( g(y) : [0, 1] \to \tilde{C} \subset [0, 1] \), strictly increasing on \([0, 1]\), which is the inverse of \( f : \tilde{C} \to [0, 1] \). For any \( a \in [0, 1] \) the set

\[ E_a := \{ y \in [0, 1] : g(y) \geq a \} = \{ y \in [0, 1] : y \geq f(a) \} \]

is measurable. Hence \( g \) is a measurable function on \([0, 1]\).

Let \( A \subset [0, 1] \) be a nonmeasurable set. Its image under \( g \) has measure zero, hence measurable. Let

\[ \varphi = \chi_{g(A)} : [0, 1] \to \mathbb{R} \]

then \( \varphi \) is a measurable function on \([0, 1]\). But the composite function \( \varphi \circ g : [0, 1] \to \mathbb{R} \) is not measurable since the set

\[ \frac{3}{2} \leq y \in [0, 1] : \varphi(g(y)) > \frac{1}{2} = A \]

is not measurable.

**Second solution (much easier):**

Let \( f(x) : [0, 1] \to [0, 1] \) be the Cantor-Lebesgue function and let \( g(x) = x + f(x) \). It is easy to see that \( g(x) : [0, 1] \to [0, 2] \) is a strictly increasing continuous function. Hence \( g(x) \) is a homeomorphism of \([0, 1]\) onto \([0, 2]\). We denote its continuous inverse by \( h : [0, 2] \to [0, 1] \). On each interval \( I_1, I_2, I_3, \ldots \), removed in the construction of the Cantor set, say the interval
\[ I_1 = \left[ \frac{1}{2}, \frac{2}{3} \right], \] the function \( g(x) \) becomes \( g(x) = x + \frac{1}{2} \). Hence \( g(x) \) sends \( I_1 \) onto an open interval with the same length. Using this observation one can see that

\[
\begin{align*}
\bar{\mathbb{A}} & = \mathbb{A} \cup \cdots \cup \mathbb{A} \\
g(I_k) & = \mathbb{I} \cup \cdots \cup \mathbb{I} \\
|g(I_k)| & = |I_k| = 1
\end{align*}
\]

which implies \(|g(C)| = 2 - 1 = 1\), where \( C \) is the Cantor set. Since \( g(C) \) has positive measure, by Corollary 3.39 in the book, there exists a non-measurable set \( B \subset g(C) \). Now consider the set \( A = h(B) \subset C \). It has measure zero, hence \( A \) is measurable. Let

\[ \varphi = \kappa_A : [0, 1] \to \mathbb{R}, \quad |A| = 0 \]

then \( \varphi \) is a measurable function on \([0, 1]\). We now have two measurable functions \( h : [0, 2] \to [0, 1] \) (continuous) and \( \varphi : [0, 1] \to \mathbb{R} \). But the composite function \( \varphi \circ h : [0, 2] \to \mathbb{R} \) is not measurable since the set

\[ \theta \in [0, 2] : \varphi(h(\theta)) > \frac{1}{2} \]

is not measurable. \( \blacksquare \)