Real Analysis Homework 3, due 2007-10-3 in class

Show Your Work to Each Problem

1. (20 points) Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a continuous function. Define the collection of sets \( P \) as

\[
X = \bigcap_{B \subset \mathbb{R} : f^{-1}(B) \text{ is measurable}}^a
\]

Does \( P \) form a \( \sigma \)-algebra? If \( B \subset \mathbb{R} \) is a Borel set, does it follow that \( f^{-1}(B) \) is a Borel set? Give your reasons.

2. (20 points) Do Exercise 12 in p. 48. Hint: You can use Theorem 3.29.

3. (10 points) It has been proved in class that if \( E \subset \mathbb{R}^n \) is an arbitrary measurable set (\(|E| = \infty \) is allowed). We have

\[
|E| = \inf_{G \supset E, G \text{ open in } \mathbb{R}^n} |G| = \sup_{F \subset E, F \text{ closed in } \mathbb{R}^n} |F|.
\]

Show that if \( E \subset \mathbb{R}^n \) is an arbitrary set satisfying the following

\[
|E|_e < \infty
\]

\[
\inf_{G \supset E, G \text{ open in } \mathbb{R}^n} |G| = \sup_{F \subset E, F \text{ closed in } \mathbb{R}^n} |F|
\]

then \( E \) must be measurable. Use an example to explain that the condition \( |E|_e < \infty \) is necessary. That is, there exists a set \( E \) with \( |E|_e = \infty \), \( \inf_{G \supset E, G \text{ open in } \mathbb{R}^n} |G| = \sup_{F \subset E, F \text{ closed in } \mathbb{R}^n} |F| \), but it is not measurable.