Real Analysis Homework 13, due 2007-12-26 in class

1. (10 points) Do Exercise 21 in P. 144.

2. (10 points) Prove the following more general version of the Tchebyshev inequality: Assume $f \geq 0$ is measurable on $E$ satisfying $\int_E f^p \, dx < \infty$, where $0 < p < \infty$ is a constant. Then for any $\alpha > 0$ we have

$$|\{x \in E : f(x) > \alpha\}| \leq \frac{1}{\alpha^p} \cdot \int_E f^p \, dx.$$

3. (10 points) Let $H(x)$ be the Heaviside function given by

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Find the set of those $x \in \mathbb{R}$ such that

$$\lim_{h \to 0^+} \frac{1}{2h} \int_{x-h}^{x+h} f(\theta) \, d\theta = f(x).$$

Determine if the point $x = 0$ is in the Lebesgue set of $f$ or not.

4. (10 points) Let $g(x)$ be the function given by

$$g(x) = \begin{cases} \sin \frac{x}{2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine if the point $x = 0$ is in the Lebesgue set of $g$ or not. Also let $G(x) = \int_0^x g(s) \, ds$, $x \in \mathbb{R}$. Do we have $G'(0) = g(0)$ or not.