Real Analysis Homework 12, due 2007-12-12 in class

1. (10 points) Do Exercise 1 in p. 123.

Solution: Assume \( |f| > 0 \) on a set \( f \in \mathbb{R}^n \) with \( |E| > 0 \). We know that there exists \( N \in \mathbb{N} \) such that \( |f| > \frac{1}{N} \) on \( E_N \). Denote the set \( |f| > \frac{1}{N} \) by \( E_N \). We have \( \frac{1}{N} \chi_{E_N}(x) \leq |f(x)| \) on \( E_N \) (and on \( \mathbb{R}^n \) also) so

\[
\frac{1}{N} \chi_{E_N}(x) \leq f^*(x) \quad \text{on} \quad \mathbb{R}^n.
\]

Hence by estimate (7.7) in p. 104, there exists a large number \( d \) such that

\[
\frac{\mu}{c_1|E_N|} \frac{1}{|x|^n} \leq \frac{1}{N} \chi_{E_N}(x) \leq f^*(x) \quad \text{for all} \quad |x| \geq d.
\]

On the compact set \( S = \{x \in \mathbb{R}^n : 1 \leq |x| \leq d \} \), since \( f^*(x) \) is lower semicontinuous on \( \mathbb{R}^n \) with \( f^*(x) > 0 \) everywhere, it attains its positive minimum on \( S \) (see exercise 7, p. 61). Hence there exists a small positive constant \( c_2 \) such that \( f^*(x) \geq \frac{c_2}{|x|^n} \) on \( S \). Let \( c = \min \frac{c_1|E_N|}{N}, c_2 \). We have

\[
f^*(x) \geq \frac{c}{|x|^n} \quad \text{for all} \quad |x| \geq 1.
\]

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2. (10 points) Do Exercise 2 in p. 123.

Solution: Assume \( |\phi| \leq M \) on \( \mathbb{R}^n \) and \( x \) is in the Lebesgue set of \( f \). We have

\[
Z \mathbb{R}^n |(f * \phi_\varepsilon)(x) - f(x)| \leq Z \mathbb{B}_\varepsilon(O) |f(x - y) - f(x)| |\phi_\varepsilon(y)| dy \\
= Z \mathbb{B}_\varepsilon(O) |f(x - y) - f(x)| |\phi_\varepsilon(y)| dy \leq M \frac{1}{\varepsilon^n} \mathbb{B}_\varepsilon(O) |f(x - y) - f(x)| dy
\]

where \( \mathbb{B}_\varepsilon(O) = \{ |y| \leq \varepsilon \} \) has measure \( C(n) \varepsilon^n \). Hence

\[
|(f * \phi_\varepsilon)(x) - f(x)| \leq C(n) M \frac{1}{\varepsilon^n} |f(x - y) - f(x)| dy
\]

and we know that

\[
\lim_{\varepsilon \to 0} \frac{1}{\mathbb{B}_\varepsilon(O)} \mathbb{B}_\varepsilon(O) |f(x - y) - f(x)| dy = 0
\]

due to Theorem 7.16.

3. (10 points) Let \( C_0(\mathbb{R}^n) \) be the space of all continuous functions on \( \mathbb{R}^n \) with compact support. We know that it is dense in the space \( L^1(\mathbb{R}^n) \) (Lemma 7.3 of the book). It is also clear that each \( g(x) \in C_0(\mathbb{R}^n) \) is uniformly continuous on \( \mathbb{R}^n \). Use this dense property to show that if \( f \in L^1(\mathbb{R}^n) \), then we have the following property called “Continuity of Translation in \( L^1 \) :

\[
\lim_{y \to 0} \int_{\mathbb{R}^n} |f(x + y) - f(x)| dx = 0.
\]
Solution: For any \( \varepsilon > 0 \), choose a function \( g(x) \in C_0(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} |f(x) - g(x)| \, dx < \frac{\varepsilon}{3} \). Then for any \( y \in \mathbb{R}^n \) we also have
\[
\int_{\mathbb{R}^n} |f(x + y) - g(x + y)| \, dx < \frac{\varepsilon}{3}.
\]
Let \( B_r(O) \) be the ball centered at the origin with large radius \( r > 0 \) such that it contains the support of \( g \). Then \( g \) is uniformly continuous on \( B_r(O) \) and so there exists \( 0 < \delta < 1 \) such that if \( |y| \leq \delta \), then
\[
|g(x + y) - g(x)| \leq \frac{\varepsilon}{|B_{r+1}(O)|} \text{ for all } x \in \mathbb{R}^n.
\]
Now if \( |y| < \delta < 1 \), we have
\[
\begin{align*}
Z & \int_{\mathbb{R}^n} |f(x + y) - f(x)| \, dx \\
& \leq \frac{\varepsilon}{3} + \int_{B_{r+1}(O)} |g(x + y) - g(x)| \, dx + \int_{\mathbb{R}^n \setminus B_{r+1}(O)} |g(x + y) - g(x)| \, dx \\
& \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.
\end{align*}
\]

4. (10 points) There are many applications of the use of convolution in analysis. One easy example is the following. Let
\[
h(t) = \begin{cases} 
1/2 & \text{if } t \leq 0 \\
v(t) & \text{if } t > 0.
\end{cases}
\]
It is known that \( h(t) \) is a \( C^\infty \) function on \( \mathbb{R} \). Next let \( g(x) = h \cdot 1 - |x|^2 \), \( x \in \mathbb{R}^n \), then \( g(x) \in C_0^\infty(\mathbb{R}^n) \). One can divide it by its integral over \( \mathbb{R}^n \) so that the new function \( \varphi(x) = g(x) / \int_{\mathbb{R}^n} g(x) \, dx = 1 \). For any number \( \varepsilon > 0 \), let \( \varphi_r(x) = \frac{1}{r^n} \varphi \left( \frac{x}{r} \right) \). Then it satisfies \( \varphi_r(x) \in C_0^\infty(\mathbb{R}^n), \varphi_r(x) \geq 0, \varphi_r(x) > 0 \iff |x| < \varepsilon, \int_{\mathbb{R}^n} \varphi_r(x) \, dx = 1 \). Show that:

(a) If \( f \in C(\mathbb{R}^n) \), then \( (f \ast \varphi_r)(x) \) converges uniformly to \( f(x) \) on compact subsets of \( \mathbb{R}^n \) as \( r \to 0^+ \). (It is easy to see that \( (f \ast \varphi_r)(x) \in C^\infty(\mathbb{R}^n) \). You do not have to show this.)

(b) If \( f \in C_0(\mathbb{R}^n) \), then for any \( \varepsilon > 0 \), \( (f \ast \varphi_r)(x) \) also has compact support.

Solution: (a) We know that \( f \) is uniformly continuous on compact subsets of \( \mathbb{R}^n \). Let \( S \) be a compact subset of \( \mathbb{R}^n \). For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[
|f(x) - f(x)| \leq \varepsilon
\]
for all \( x \in S, |y| \leq \delta \).
For \( x \in S \) we have
\[
\begin{align*}
|((f \ast \varphi_\delta)(x)) - f(x)| &= \int_{\mathbb{R}^n} \left[ (f(x) - f(x)) \ast \varphi_\delta(y) \right] \, dy \\
& \leq \int_{B_\delta(O)} |f(x - y) - f(x)| \cdot \varphi_\delta(y) \, dy \\
& \leq \int_{B_\delta(O)} |f(x - y) - f(x)| \cdot \varphi_\delta(y) \, dy \leq \varepsilon.
\end{align*}
\]
where \( B_\delta(O) = \{ y | |y| \leq \delta \} \). Hence \( (f \ast \varphi_\delta)(x) \) converges uniformly to \( f(x) \) on \( S \) as \( \delta \to 0^+ \).
(b) Assume $S$ is the compact support of $f$. Then
\[
(f * \varphi_\varepsilon)(x) = \int_{\mathbb{R}^n} f(x - y) \cdot \varphi_\varepsilon(y) \, dy = \int_{B_\varepsilon(O)} f(x - y) \cdot \varphi_\varepsilon(y) \, dy.
\]
From above we see that if $x \notin S$ with $\text{dist}(x, S) > \varepsilon$, then we also have $x - y \notin S$ for any $y \in B_\varepsilon(O)$.
For such $x$, we have
\[
(f * \varphi_\varepsilon)(x) = \int_{B_\varepsilon(O)} f(x - y) \cdot \varphi_\varepsilon(y) \, dy = 0.
\]
Hence $(f * \varphi_\varepsilon)(x)$ also has compact support. \hfill \Box