Fractional Order Riemann-Liouville Integral Equations with Multiple Time Delays

Saïd Abbas\(^1\), Mouffak Benchohra\(^2\)

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Abstract
In the present article we investigate the existence and uniqueness of solutions for a system of integral equations of fractional order by using some fixed point theorems. Also we illustrate our results with some examples.

1 Introduction
The idea of fractional calculus and fractional order integral equations has been a subject of interest not only among mathematicians, but also among physicists and engineers. Indeed, we can find numerous applications in rheology, control, porous media, viscoelasticity, electrochemistry, electromagnetism, etc. [9, 11, 16, 17, 19]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas et al. [14], Miller and Ross [18], Samko et al. [21], the papers of Abbas and Benchohra [1, 2], Abbas et al. [3], Belarbi et al. [4], Benchohra et al. [5, 6, 7], Diethelm [8], Kilbas and Marzan [15], Mainardi [16], Podlubny et al. [20], Vityuk [22], Vityuk and Golushkov [23], and Zhang [24] and the references therein.

In [13], R. W. Ibrahim and H. A. Jalab studied the existence of solutions of the following fractional integral inclusion

\[ u(t) - \sum_{i=1}^{m} b_i(t)u(t - \tau_i) \in \int_0^t F(t, u(t)) \text{ if } t \in [0, T], \]

where \( \tau_i < t \in [0, T], b_i : [0, T] \to \mathbb{R}, i = 1, \ldots, n \) are continuous functions, and \( F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) is a given multivalued map.

This paper concerned with the existence and uniqueness of solutions for the following fractional order integral equations for the system

\[ u(x, y) = \sum_{i=1}^{m} g_i(x, y)u(x-\xi_i, y-\mu_i)+\int_0^y f(x, y, u(x, y)) \text{ if } (x, y) \in J := [0, a] \times [0, b], \quad (1) \]

*Mathematics Subject Classifications: 26A33
\(^1\)Laboratoire de Mathématiques, Université de Saida, B.P. 138, 20000, Saida, Algérie
\(^2\)Laboratoire de Mathématiques, Université de Sidi Bel-Abbès, B.P. 89, 22000, Sidi Bel-Abbès, Algérie
where $a, b > 0$, $\theta = (0, 0)$, $\xi_i, \mu_i \geq 0$; $i = 1, \ldots, m$, $\xi = \max_{i=1,\ldots,m}\{\xi_i\}$, $\mu = \max_{i=1,\ldots,m}\{\mu_i\}$, $I^\alpha_\beta$ is the left-sided mixed Riemann-Liouville integral of order $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $g_i : J \to \mathbb{R}$; $i = 1, \ldots, m$ are given continuous functions, and $\Phi : \tilde{J} \to \mathbb{R}^n$ is a given continuous function such that

\[
\Phi(x, 0) = \sum_{i=1}^{m} g_i(x, 0)\Phi(x - \xi_i, -\mu_i); \ x \in [0, a],
\]

and

\[
\Phi(0, y) = \sum_{i=1}^{m} g_i(0, y)\Phi(-\xi_i, y - \mu_i); \ y \in [0, b].
\]

We present three results for the problem (1)-(2), the first one is based on Schauder’s fixed point theorem (Theorem 1), the second one is a uniqueness of the solution by using the Banach fixed point theorem (Theorem 2) and the last one on the nonlinear alternative of Leray-Schauder type (Theorem 4).

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J)$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}^n$ with the norm

\[
\|w\|_\infty = \sup_{(x,y) \in J} \|w(x,y)\|,
\]

where $\|\cdot\|$ denotes a suitable complete norm on $\mathbb{R}^n$. Also, $C := C([-\xi, a] \times [-\mu, b])$ is a Banach space endowed with the norm

\[
\|w\|_C = \sup_{(x,y) \in [-\xi, a] \times [-\mu, b]} \|w(x,y)\|.
\]

As usual, by $L^1(J)$ we denote the space of Lebesgue-integrable functions $w : J \to \mathbb{R}^n$ with the norm

\[
\|w\|_{L^1} = \int_0^a \int_0^b \|w(x,y)\| \, dy \, dx.
\]

DEFINITION 1 ([23]). Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, $\theta = (0, 0)$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order $r$ of $u$ is defined by

\[
(I^\alpha_\beta u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} u(s,t) \, dtds.
\]

In particular,

\[
(I^\alpha_\beta u)(x, y) = u(x,y), \quad (I^\alpha_\beta u)(x, y) = \int_0^x \int_0^y u(s,t) \, dt \, ds \text{ for almost all } (x, y) \in J,
\]
where $\sigma = (1, 1)$.
For instance, $I_{\sigma}^u$ exists for all $r_1, r_2 \in (0, \infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I_{\sigma}^u) \in C(J)$, moreover

$$(I_{\sigma}^u)(x, 0) = (I_{\sigma}^u)(0, y) = 0; \; x \in [0, a], \; y \in [0, b].$$

**EXAMPLE 1.** Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_{\sigma} x^\lambda y^\omega = \frac{\Gamma(1 + \lambda) \Gamma(1 + \omega)}{\Gamma(1 + \lambda + r_1) \Gamma(1 + \omega + r_2)} x^{\lambda + r_1} y^{\omega + r_2}$$
for almost all $(x, y) \in J$.

### 3 Existence of Solutions

Let us start by defining what we mean by a solution of the problem (1)-(2).

**DEFINITION 2.** A function $u \in C$ is said to be a solution of (1)-(2) if $u$ satisfies equation (1) on $J$ and condition (2) on $\tilde{J}$.

Set

$$B = \max_{i=1, \ldots, m} \left\{ \sup_{(x, y) \in J} |g_i(x, y)| \right\}.$$

**THEOREM 1.** Assume

($H_1$) There exists a positive function $h \in C(J)$ such that

$$\|f(x, y, u)\| \leq h(x, y), \text{ for all } (x, y) \in J \text{ and } u \in \mathbb{R}^n.$$

If $mB < 1$, then problem (1)-(2) has at least one solution $u$ on $[\xi, a] \times [-\mu, b]$.

**PROOF.** Transform problem (1)-(2) into a fixed point problem. Consider the operator $N : C \to C$ defined by,

$$N(u)(x, y) = \begin{cases} \Phi(x, y); & (x, y) \in \tilde{J} , \\ \sum_{i=1}^m g_i(x, y)u(x - \xi_i, y - \mu_i) + I_{\sigma} f(x, y, u(x, y)); & (x, y) \in J. \end{cases} \quad (3)$$

The problem of finding the solutions of problem (1)-(2) is reduced to finding the solutions of the operator equation $N(u) = u$. Let

$$R = \frac{R^*}{1-mB}$$
where

$$R^* = \frac{a^{r_1} b^{r_2} h^*}{\Gamma(1 + r_1) \Gamma(1 + r_2)},$$
and $h^* = \|h\|_{\infty}$, and consider the set

$$B_R = \{ u \in C : \|u\|_C \leq R \}.$$
It is clear that $B_R$ is a closed bounded and convex subset of $C$. For every $u \in B_R$ and $(x, y) \in \tilde{J}$ we obtain by $(H_1)$ that

$$
\|N(u)(x, y)\| \leq \sum_{i=1}^{m} |g_i(x, y)| \|u(x - \xi_i, y - \mu_i)\| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s, t, u(s, t))\| dt ds
$$

$$
\leq mB\|u\|_C + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} h(s, t) dt ds
$$

$$
\leq mBR + \frac{mB(1 + r_1)(1 + r_2)}{\Gamma(1 + r_1)\Gamma(1 + r_2)} a^{r_1}b^{r_2} l^*
$$

On the other hand, for every $u \in B_R$ and $(x, y) \in \tilde{J}$, we obtain

$$
\|N(u)(x, y)\| = \|\Phi(x, y)\| \leq R.
$$

So we obtain that

$$
\|N(u)\|_C \leq R.
$$

That is, $N(B_R) \subseteq B_R$. Since $f$ is bounded on $B_R$, thus $N(B_R)$ is equicontinuous and the Schauder fixed point theorem shows that $N$ has at least one fixed point $u^* \in B_R$ which is solution of (1)-(2).

For the uniqueness we prove the following Theorem

**THEOREM 2.** Assume that following hypothesis holds:

$(H_2)$ There exists a positive function $l \in C(J)$ such that

$$
\|f(x, y, u) - f(x, y, v)\| \leq l(x, y)\|u - v\|,
$$

for each $(x, y) \in J$ and $u, v \in \mathbb{R}^n$.

If

$$
mB\frac{\Gamma(1 + r_1)\Gamma(1 + r_2)}{\Gamma(1 + r_1)\Gamma(1 + r_2)} a^{r_1}b^{r_2} l^* < 1,
$$

(4)

where $l^* = \|l\|_\infty$, then problem (1)-(2) has a unique solution on $[-\xi, a] \times [-\mu, b]$.

**PROOF.** Consider the operator $N$ defined in (3). Then by $(H_2)$, for every $u, v \in C$
and \((x, y) \in J\) we have
\[
\|N(u)(x, y) - N(v)(x, y)\| \leq \sum_{i=1}^{m} g_i(x, y) \|u(x - \xi_i, y - \mu_i) - v(x - \xi_i, y)\|
\]
\[
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{0}^{x} \int_{0}^{y} (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} |f(s, t, u(s, t)) - f(s, t, v(s, t))| dt \, ds
\]
\[
\leq mB\|u - v\|_{\infty}
\]
\[
+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{0}^{x} \int_{0}^{y} (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} l(s, t) \|u - v\|_{C} dt \, ds
\]
\[
\leq mB\|u - v\|_{\infty} + l^{*} \frac{a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \|u - v\|_{C}
\]
\[
= \left( mB + \frac{l^{*}a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \right) \|u - v\|_{C}.
\]

Thus
\[
\|N(u) - N(v)\|_{C} \leq \frac{mB\Gamma(1 + r_1)\Gamma(1 + r_2) + a^{r_1}b^{r_2}l^{*}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} \|u - v\|_{C}
\]

Hence by (4), we have that \(N\) is a contraction mapping. Then in view of Banach fixed point Theorem, \(N\) has a unique fixed point which is solution of problem (1)-(2).

THEOREM 3 ([10]). (Nonlinear alternative of Leray-Schauder type) By \(\overline{U}\) and \(\partial U\) we denote the closure of \(U\) and the boundary of \(U\) respectively. Let \(X\) be a Banach space and \(C\) a nonempty convex subset of \(X\). Let \(U\) a nonempty open subset of \(C\) with \(0 \in U\) and \(T : \overline{U} \rightarrow C\) continuous and compact operator. Then either

(a) \(T\) has fixed points, or

(b) there exist \(u \in \partial U\) and \(\lambda \in (0, 1)\) with \(u = \lambda T(u)\).

In the sequel we use the following version of Gronwall’s Lemma for two independent variables and singular kernel.

LEMMA 1 ([12]). Let \(v : J \rightarrow [0, \infty)\) be a real function and \(\omega(\cdot, \cdot)\) be a nonnegative, locally integrable function on \(J\). If there are constants \(c > 0\) and \(0 < r_1, r_2 < 1\) such that
\[
v(x, y) \leq \omega(x, y) + c \int_{0}^{x} \int_{0}^{y} \frac{v(s, t)}{(x - s)^{r_1}(y - t)^{r_2}} dt \, ds,
\]
then there exists a constant \(\delta = \delta(r_1, r_2)\) such that
\[
v(x, y) \leq \omega(x, y) + \delta c \int_{0}^{x} \int_{0}^{y} \frac{\omega(s, t)}{(x - s)^{r_1}(y - t)^{r_2}} dt \, ds,
\]
for every \((x, y) \in J\).

Now, we present an existence result for the problem (1)-(2) based on the Nonlinear alternative of Leray-Schauder type.

THEOREM 4. Assume
(H3) There exist positive functions $p, q \in C(J)$ such that
\[
\|f(x, y, u)\| \leq p(x, y) + q(x, y)\|u\|, \text{ for all } (x, y) \in J \text{ and } u \in \mathbb{R}^n.
\]
If $mB < 1$, then problem (1)-(2) has at least one solution on $[-\xi, a] \times [-\mu, b]$.

PROOF. Consider the operator $N$ defined in (3). We shall show that the operator $N$ is completely continuous. By the continuity of $f$ and the Arzela-Ascoli Theorem, we can easily obtain that $N$ is completely continuous.

A priori bounds. We shall show there exists an open set $U \subseteq C$ with $u \neq \lambda N(u)$, for $\lambda \in (0, 1)$ and $u \in \partial U$. Let $u \in C$ and $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x, y) \in J$, we have
\[
u(x, y) = \lambda \sum_{i=1}^{m} g_{i}(x, y)u(x - \xi_{i}, y - \mu_{i}) + \lambda \mu_{0}f(x, y, u(x, y)).
\]
This implies by (H3) that, for each $(x, y) \in J$, we have
\[
\|u(x, y)\| \leq mB\|u(x, y)\| + \frac{p^*a^{r_1}b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} + \frac{q^*}{\Gamma(r_1)\Gamma(r_2)} \int_{0}^{y} \int_{0}^{y} (x - s)^{r_1 - 1}(y - t)^{r_2 - 1}u(s, t)dt\,
ds,
\]
where $p^* = \|p\|_{\infty}$ and $q^* = \|q\|_{\infty}$. Thus, for each $(x, y) \in J$, we get
\[
\|u(x, y)\| \leq \frac{p^*a^{r_1}b^{r_2}}{(1 - mB)\Gamma(1 + r_1)\Gamma(1 + r_2)} + \frac{q^*}{(1 - mB)\Gamma(r_1)\Gamma(r_2)} \int_{0}^{y} \int_{0}^{y} (x - s)^{r_1 - 1}(y - t)^{r_2 - 1}u(s, t)dt\,
ds
\leq w + c \int_{0}^{y} \int_{0}^{y} (x - s)^{r_1 - 1}(y - t)^{r_2 - 1}u(s, t)dt\,
ds,
\]
where
\[
w := \frac{p^*a^{r_1}b^{r_2}}{(1 - mB)\Gamma(1 + r_1)\Gamma(1 + r_2)},
\]
and
\[
c := \frac{q^*}{(1 - mB)\Gamma(r_1)\Gamma(r_2)}.
\]
From Lemma 1, there exists $\delta := \delta(r_1, r_2) > 0$ such that, for each $(x, y) \in J$, we get
\[
\|u\|_{\infty} \leq w \left(1 + c\delta \int_{0}^{y} \int_{0}^{y} (x - s)^{r_1 - 1}(y - t)^{r_2 - 1}dt\,
ds\right)
\leq w \left(1 + \frac{c\delta a^{r_1}b^{r_2}}{r_1 r_2}\right) := \tilde{M}.
\]
Set $M^* := \max\{\|\Phi\|, \tilde{M}\}$ and
\[
U = \{u \in C : \|u\|_{C} < M^* + 1\}.
\]
By our choice of $U$, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$. As a consequence of Theorem 3, we deduce that $N$ has a fixed point $u$ in $\overline{U}$ which is a solution to problem (1)-(2).

4 Examples

We provide two examples.

EXAMPLE 1. As an application of our results we consider the following system of fractional integral equations of the form

\[ u(x, y) = \frac{x^3 y}{8} u(x - \frac{3}{4}, y - 3) + \frac{x^4 y^2}{12} u(x - 2, y - \frac{1}{2}) + \frac{1}{4} u(x - 1, y - \frac{3}{2}) + I_0^a f(x, y, u); \quad \text{if} \quad (x, y) \in J := [0, 1] \times [0, 1], \]

\[ u(x, y) = 0; \quad \text{if} \quad (x, y) \in \tilde{J} := [-2, 1] \times [-3, 1] \setminus [0, 1] \times (0, 1], \]

where $m = 3$, $r = (\frac{1}{2}, \frac{1}{2})$ and

\[ f(x, y, u) = e^{x+y} \frac{1}{1 + |u|}. \]

Set

\[ g_1(x, y) = \frac{x^3 y}{8}, \quad g_2(x, y) = \frac{x^4 y^2}{12}, \quad g_3(x, y) = \frac{1}{4}. \]

We have $B = \frac{1}{4}$ and

\[ |f(x, y, u)| \leq e^{x+y}; \quad \text{for all} \quad (x, y) \in J \quad \text{and} \quad u \in \mathbb{R}. \]

Then condition $(H_1)$ is satisfied and $mB = \frac{3}{4} < 1$. In view of Theorem 1, problem (5)-(6) has a solution defined on $[-2, 1] \times [-3, 1]$.

EXAMPLE 2. Consider the fractional integral equation

\[ u(x, y) = \frac{x^3 y}{8} u(x - 1, y - \frac{1}{2}) + \frac{x^4 y^2}{12} u(x - 2, y - \frac{3}{4}) + \frac{1}{8} u(x - 3, y - 2) + I_0^a f(x, y, u); \quad \text{if} \quad (x, y) \in J := [0, 1] \times [0, 1], \]

\[ u(x, y) = \Phi(x, y); \quad \text{if} \quad (x, y) \in \tilde{J} := [-3, 1] \times [-2, 1] \setminus [0, 1] \times (0, 1], \]

where $m = 3$, $r = (\frac{1}{2}, \frac{1}{2})$, $f(x, y, u) = \frac{x^3 y^2}{20} \frac{|u|}{1 + |u|}$ and $\Phi : \tilde{J} \to \mathbb{R}$ is continuous with

\[ \Phi(x, 0) = \frac{1}{8} \Phi(x - 3, -2), \quad \Phi(0, y) = \frac{1}{8} \Phi(-3, y - 2); \quad x, y \in [0, 1]. \]

Notice that condition (9) is satisfied by $\Phi \equiv 0$.

Set

\[ g_1(x, y) = \frac{x^3 y}{8}, \quad g_2(x, y) = \frac{x^4 y^2}{12}, \quad g_3(x, y) = \frac{1}{4}. \]

We have $B = \frac{1}{4}$. It is clear that $f$ satisfies $(H_2)$ with $l^* = \frac{1}{10}$. A simple computation shows that condition (4) is satisfied. Hence by Theorem 2, problem (7)-(8) has a unique solution defined on $[-3, 1] \times [-2, 1]$.

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References


