A Characterization Of A Family Of Semiclassical Orthogonal Polynomials Of Class One*

Mohamed Ihsen Tounsi†

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Abstract
In this paper, we give another characterization of a non-symmetric semiclassical orthogonal polynomials of class one.

1 Introduction
Our goal is to characterize the set of non-symmetric semiclassical orthogonal polynomials of class one \( \{W_n\}_{n \geq 0} \) verifying the three-term recurrence relation with \( \beta_n = (-1)^n \), \( n \geq 0 \) in a concise way as in [5, 6] via the study of the functional equation \((\Phi w)' + \Psi w = 0\) satisfied by its corresponding regular form \( w \). Some information about the shape of polynomials \( \Phi \) and \( \Psi \) intervening in the above functional equation are given due to the quadratic decomposition of \( \{W_n\}_{n \geq 0} \) and to a connection between \( w \) and a suitable symmetric regular form \( \vartheta \). As application, we characterize \( w \) by giving the functional equation, the recurrence coefficient \( \gamma_{n+1}, n \geq 0 \) and an integral representation.

We denote by \( P \) the vector space of polynomials with coefficients in \( \mathbb{C} \) and by \( P' \) its dual space. The action of \( u \in P' \) on \( f \in P \) is denoted as \( \langle u, f \rangle \). In particular, we denote by \( (u)_n := \langle u, x^n \rangle \), \( n \geq 0 \), the moments of \( u \). For instance, for any form \( u \), any polynomial \( g \) and any \( (a, b, c) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^2 \), we let \( Du = u' \), \( gu = u \), \( h_\alpha u \), \( (x - c)^{-1} u \) and \( \delta_c \), be the forms defined in [3]:

\[
\langle u', f \rangle := -\langle u, f' \rangle, \quad \langle \sigma u, f \rangle := \langle u, \sigma f \rangle, \quad \langle gu, f \rangle := \langle u, gf \rangle, \quad \langle h_\alpha u, f \rangle := \langle u, h_\alpha f \rangle,
\]

\[
\langle \tau_{\alpha} u, f \rangle := \langle u, \tau_{-\alpha} f \rangle, \quad \langle (x - c)^{-1} u, f \rangle := \langle u, \delta_c f \rangle, \quad \langle \delta_c, f \rangle := \langle \delta_c, f \rangle,
\]

where \( \langle \sigma f \rangle (x) = f(x^2) \), \( \langle h_\alpha f \rangle (x) = f(ax) \), \( \langle \tau_{-\alpha} f \rangle (x) = f(x + b) \), \( \langle \delta_c, f \rangle (x) = \frac{f(x) - f(c)}{x - c} \)

for all \( f \in P \). It is easy to see that [3, 4]

\[
(fu)' = fu' + f'u, \quad f \in P, \quad u \in P', \tag{1}
\]

\[
f(x)\sigma u = \sigma(f(x)u), \quad f \in P, \quad u \in P', \tag{2}
\]

\[
\sigma(u') = 2(\sigma(xu))', \quad u \in P', \tag{3}
\]

*Mathematics Subject Classification: 33C45, 42C05.
†Institut Supérieur des Sciences Appliquées et de Technologie, University of Gabès, Rue Omar Ibn El Khattab, 6072, Gabès, Tunisia

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\[ x^{-1}(xu) = u - (u_0)\delta, \quad x(x^{-1}u) = u, \quad u \in \mathcal{D}'. \quad (4) \]

A form \( w \) is said to be regular whenever there is a sequence of monic polynomials \( \{W_n\}_{n \geq 0}, \deg W_n = n, n \geq 0 \) (MPS) such that \( \langle w, W_nW_m \rangle = k_n\delta_{n,m}, n, m \geq 0 \) with \( k_n \neq 0 \) for any \( n \geq 0 \). In this case, \( \{W_n\}_{n \geq 0} \) is called a monic orthogonal polynomial sequence (MOPS) and it is characterized by the following three-term recurrence relation [1]

\[
W_0(x) = 1, \quad W_1(x) = x - \beta_0, \\
W_{n+2}(x) = (x - \beta_{n+1})W_{n+1}(x) - \gamma_{n+1}W_n(x), \quad n \geq 0,
\]

where \( \beta_n = \frac{\langle w, xW_n^2 \rangle}{\langle w, W_n^2 \rangle} \in \mathbb{C} \) and \( \gamma_{n+1} = \frac{\langle w, W_{n+1}^2 \rangle}{\langle w, W_n^2 \rangle} \in \mathbb{C} \setminus \{0\}, \ n \geq 0. \)

When \( w \) is regular, \( \{W_n\}_{n \geq 0} \) is a symmetric (MOPS) if and only if \( \beta_n = 0, \ n \geq 0 \) or equivalently \( \langle w, W_{n+1} \rangle = 0, \ n \geq 0 \). Also, The form \( w \) is said to be normalized if \( \langle w, 1 \rangle = 1 \). In this paper, we suppose that any form will be normalized.

A form \( w \) is called semiclassical when it is regular and there exist two polynomials \( \Phi, \Psi, \deg \Phi = t \geq 0, \deg \Psi = p \geq 1 \) such that

\[
(\Phi w)' + \Psi w = 0. \quad (6)
\]

It’s corresponding orthogonal polynomial sequence \( \{W_n\}_{n \geq 0} \) is called semiclassical. The semiclassical character is kept by shifting [3, 4, 5]. In fact, let \( \{a^{-n}W_n(ax+b)\}_{n \geq 0}, \ a \neq 0, \ b \in \mathbb{C}; \) when \( w \) satisfies (6), then \( \left(h_{a^{-1}} \circ \tau_{-b}\right)w \) fulfills

\[
(a^{-t}\Phi(ax+b)(h_{a^{-1}} \circ \tau_{-b})w)' + a^{1-t}\Psi(ax+b)(h_{a^{-1}} \circ \tau_{-b})w = 0, \quad (7)
\]

and the recurrence coefficients of (5) are

\[
\frac{\beta_n - b}{a}, \quad \frac{\gamma_{n+1}}{a^2}, \quad n \geq 0. \quad (8)
\]

The semiclassical form \( w \) is said to be of class \( s = \max(p - 1, t - 2) \geq 0 \) if and only if [3, 4, 5]

\[
\prod_{c \in \mathcal{Z}_\Phi} \left\{ (\Psi(c) + \Phi'(c)) + \left( \langle w, (\theta_c \Psi) + (\theta_c^2 \Phi) \rangle \right) \right\} > 0, \quad (9)
\]

where \( \mathcal{Z}_\Phi \) is the set of zeros of \( \Phi \). In particular, when \( s = 0 \) the form \( w \) is usually called classical Hermite, Laguerre, Bessel and Jacobi, see [3, 4, 5].

**LEMMA 1 ([3]).** Let \( w \) be a symmetric semiclassical form of class \( s \) satisfying (6). The following statements hold.

i) When \( s \) is odd then the polynomial \( \Phi \) is odd and \( \Psi \) is even.

ii) When \( s \) is even then the polynomial \( \Phi \) is even and \( \Psi \) is odd.

Let \( \{W_n\}_{n \geq 0} \) be a (MOPS) with respect to the form \( w \) fulfilling the three-term recurrence relation (5) with

\[
\beta_n = (-1)^n, \quad n \geq 0. \quad (10)
\]
Such a (MOPS) is characterized by the following quadratic decomposition \[4\]
\[
W_{2n}(x) = P_n(x^2), \quad W_{2n+1}(x) = (x-1)P_n^*(x^2), \quad n \geq 0,
\]
where \(\{P_n\}_{n \geq 0}\) is a (MOPS) and \(\{P_n^*\}_{n \geq 0}\) is the sequence of monic Kernel polynomials of \(K\)-parameter 1 associated with \(\{P_n\}_{n \geq 0}\) defined by [1, 2]
\[
P_n^*(x) = \frac{1}{x-1} \left[ P_{n+1}(x) - \frac{P_{n+1}(1)}{P_n(1)} P_n(x) \right], \quad n \geq 0.
\]
Furthermore the sequences \(\{P_n\}_{n \geq 0}\) and \(\{P_n^*\}_{n \geq 0}\) satisfy respectively the recurrence relation (5) with
\[
\begin{cases}
\beta_0^P = \gamma_1 + 1,
\beta_{n+1}^P = \gamma_{2n+2} + \gamma_{2n+3} + 1,
\gamma_{n+1}^P = \gamma_{2n+1} \gamma_{2n+2},
\end{cases}
\begin{cases}
\beta_0^s = \gamma_1 + \gamma_2 + 1,
\beta_{n+1}^s = \gamma_{2n+3} + \gamma_{2n+4} + 1,
\gamma_{n+1}^s = \gamma_{2n+2} \gamma_{2n+3},
\end{cases}
\]
for all \(n \geq 0\). Denoting by \(u\) and \(v\) the forms associated with \(\{P_n\}_{n \geq 0}\) and \(\{P_n^*\}_{n \geq 0}\) respectively, we get [4]
\[
u = \gamma_1^{-1}(x-1)\sigma w.
\]
The regularity of \(v\) means that [1]
\[
P_{n+1}(1) \neq 0, \quad n \geq 0.
\]
Moreover, the form \((x-1)w\) is antisymmetric, that is,
\[
((x-1)w)_{2n} = 0, \quad n \geq 0.
\]
Let now \(\lambda\) be a non-zero complex number and \(\vartheta\) be the form such that
\[
\lambda x \vartheta = (x-1)w.
\]
According to (17)-(18) we get \((x\vartheta)_{2n} = 0, \quad n \geq 0\). Hence \(\vartheta\) is a symmetric form. Multiplying (18) by \(x\), applying the operator \(\sigma\) and using (15) we get \(\lambda x \sigma \vartheta = \gamma_1 v\). Consequently, according to [3], the form \(\vartheta\) is regular if and only if
\[
\Omega_n(\lambda) = \gamma_1 P_{n-1}^{(1)}(0) + \lambda P_n^*(0) \neq 0, \quad n \geq 0,
\]
with \(P_n^{(1)}(x) = (\psi_0 P_{n+1}^*)(x), \quad n \geq 0\) and \(P_{-1}^{(1)}(x) := 0\).

LEMMA 2. There exists a non zero constant \(\lambda\) such that the form \(\vartheta\) given by (18) is regular.

PROOF. According to the following relation [2]
\[
P_{n+1}^{(1)}(x)P_{n+1}^*(x) - P_n^{(1)}(x)P_{n+2}^*(x) = \prod_{\nu=0}^{n} \gamma_{\nu+1}^s \neq 0, \quad n \geq 0,
\]
it is easy to see that
\[ |P_n^{(1)}(0)| + |P_n^*(0)| \neq 0, \quad \forall n \geq 0. \tag{20} \]
Let \( n \) be a fixed nonnegative integer. If \( P_n^{(1)}(0) = 0 \), then \( P_n^*(0) \neq 0 \) from (20). So, condition (19) is satisfied for \( \lambda \neq 0 \). If \( P_n^*(0) = 0 \), then \( P_n^{(1)}(0) \neq 0 \) from (20). So, condition (19) satisfied for \( \lambda \neq 0 \). If \( P_n^{(1)}(0) \neq 0 \) and \( P_n^*(0) \neq 0 \), then for all \( \lambda \neq \lambda_n \), (20) is satisfied, where we have posed
\[ \lambda_n = -\frac{P_n^{(1)}(0)}{P_n^*(0)}, \quad n \geq 0. \tag{21} \]
In any case there exists a constant \( \lambda \neq 0 \) such that (19) is fulfilled and so \( \vartheta \) is a regular form.

In what follows we assume that the (MOPS) \( \{W_n\}_{n \geq 0} \) associated with (5),(10) is semiclassical of class \( s_w \). Its corresponding regular form \( w \) is then semiclassical of class \( s_w \) satisfying the functional equation (6). Multiplying the equation (6) by \( (x - 1)^2 \) and on account of (1) and (18), we deduce that the form \( \# \), when it is regular, is also semiclassical of class \( s_w \) at most \( s_w + 2 \) satisfying the functional equation
\[ (E\vartheta')' + F\vartheta = 0, \tag{22} \]
with
\[ E(x) = x(x - 1)\Phi(x); \quad F(x) = x((x - 1)\Psi(x) - 2\Phi(x)). \tag{23} \]
The next technical lemma is needed in the sequel.

**LEMMA 3.** For all root \( c \) of \( \Phi \), we have
\[ a) \quad \left\langle \vartheta, \theta^2cE + \theta cF \right\rangle = \frac{1}{\chi}(c - 1)^2 \left\langle w, \theta c\Psi + \theta^2c\Phi \right\rangle + (1 - \frac{1}{\chi})(c - 1)(\Phi'(c) + \Psi(c)), \]
\[ b) \quad E'(c) + F(c) = c(c - 1)(\Phi'(c) + \Psi(c)). \tag{24} \]

**PROOF.** Let \( c \) be a root of \( \Phi \). Write \( \Phi(x) = (x - c)\Phi_c(x) \) with \( \Phi_c(x) = (\theta_c\Phi)(x) \). From (22)-(23) we have
\[ (\theta^2cE + \theta cF)(x) = \theta_c\left\{ \xi(\xi - 1)(\Phi_c(\xi) + \Psi(\xi)) \right\}(x) - 2x\Phi_c(x). \tag{25} \]
Taking \( g(x) = (\Phi_c + \Psi)(x) \) and \( f(x) = x(x - 1) \) in the following relation
\[ \theta_c(fg)(x) = g(x)(\theta_c f)(x) + f(c)(\theta_c g)(x), \quad \text{for all } f, g \in \mathcal{P}, \tag{26} \]
(25) becomes
\[ (\theta^2cE + \theta cF)(x) = (c - 1) \left\{ (\Phi_c + \Psi)(x) + c(\theta_c(\Phi_c + \Psi))(x) \right\} + x(\Psi - \Phi_c)(x). \tag{27} \]
From the second identity in (4), relation (18) is equivalent to
\[ \vartheta = \frac{1}{\chi}(w - x^{-1}w) + (1 - \frac{1}{\chi})\delta_0. \tag{28} \]
We may also write
\[
\frac{1}{\lambda} \langle w, x^{-1} w \rangle, \theta^2 E + \theta F \rangle = \frac{1}{\lambda} \langle w, \theta^2 E + \theta F - \theta_0 (\theta^2 E + \theta F) \rangle.
\] (29)

Taking \( f(x) = (\theta_c (\Phi_c + \Psi)) (x) \) in the following
\[
\frac{c}{\theta_0} (\theta_c f + \theta_0 f), \quad f \in \mathcal{P}, \quad c \in \mathbb{C},
\] (30)

and applying the operator \( \theta_0 \) to (27), we obtain
\[
(\theta_0 (\theta^2 E + \theta F)) (x) = (\Psi - \Phi_c) (x) + (c - 1) (\theta_c (\Phi_c + \Psi)) (x).
\] (31)

This gives
\[
(\theta^2 E + \theta F)(x) - (\theta_0 (\theta^2 E + \theta F)) (x) = (c - 1)^2 (\theta_c (\Phi_c + \Psi)) (x) + (x + c - 2) \Psi - \Phi.
\] (32)

Thus (29) becomes
\[
\frac{1}{\lambda} \langle w, x^{-1} w \rangle, \theta^2 E + \theta F \rangle = \frac{1}{\lambda} (c - 1)^2 \langle w, \theta_c \Phi_c + \theta_c \Psi \rangle,
\] (33)

since \( \langle w, \Psi \rangle = 0 \) and \( \langle w, x \Psi(x) - \Phi(x) \rangle = 0 \) from (6). Next, by a simple calculation, we have
\[
\left( 1 - \frac{1}{\lambda} \right) \delta_0, \theta^2 E + \theta F \right) = (1 - \frac{1}{\lambda}) (c - 1)(\Phi_c + \Psi)(c).
\] (34)

Adding (33) and (34) we obtain the first relation in (24). From (22)-(23), we have \( E'(c) = c(c - 1) \Psi'(c) \) and \( F(c) = c(c - 1) \Psi(c) \), hence the second relation in (24) holds.

Let us recall the following result about the class \( s_\theta \) of the form \( \psi \).

THEOREM 1. The form \( \psi \) is semiclassical and its class depends only on the zero \( x = 1 \) for any \( \lambda \neq \lambda_n, n \geq -1 \) where \( \lambda_n, n \geq 0 \) is given by (21) and
\[
\lambda_{-1} = \frac{\langle w, \theta_0 \Psi + \theta^2_0 \Phi \rangle + \Phi'(0) + \Psi(0)}{\Phi'(0) + \Psi(0)}.
\] (35)

Moreover, the semiclassical form \( \tilde{\psi} \) is of class \( s_\theta \) satisfying the functional equation
\[
\left( \tilde{E} \tilde{\psi} \right)' + \tilde{F} \tilde{\psi} = 0,
\] (36)

such that
\( a) \) if \( \Phi(1) \neq 0 \), then \( s_\theta = s_w + 2 \),
\[
\tilde{E}(x) = x(x - 1) \Phi(x) \quad \text{and} \quad \tilde{F}(x) = x ((x - 1) \Psi(x) - 2 \Phi(x));
\]
\( b) \) if \( \Phi(1) = 0 \) and \( \Psi(1) \neq 0 \), then \( s_\theta = s_w + 1 \),
\[
\tilde{E}(x) = x \Phi(x) \quad \text{and} \quad \tilde{F}(x) = x (\Psi(x) - (\theta_1 \Phi)(x)).
\]
c) if $\Phi(1) = 0$ and $\Psi(1) = 0$, then $s_\theta = s_w$,

$$E(x) = x(\theta_1 \Phi)(x) \quad \text{and} \quad \hat{F}(x) = x(\theta_1 \Psi)(x).$$

**PROOF.** By our assumption, on account of Lemma 2, and by (22)-(23), the form $\vartheta$ is regular and so is semiclassical of class $s_\theta \leq s_w + 2$. Let $c$ be a root of $E$ such that $c \neq 1$. According to (23) we get $c\Phi(c) = 0$. If $c \neq 0$, then $c$ is a root of $\Phi$. We suppose $E'(c) + F(c) = 0$. From (24) we obtain $E'(x) + \Psi(c) = 0$ and $\langle \vartheta, \theta_0^2 E + \theta_0 F \rangle = \frac{1}{\lambda} (c - 1)^2 \langle w, \theta_0 \Psi + \theta_0^2 \Phi \rangle \neq 0$, because $w$ is semiclassical and so satisfies (9). If $c = 0$ and $\Phi(0) \neq 0$, then $E'(0) + F(0) = -\Phi(0) \neq 0$ from (23). If $c = 0$ and $\Phi(0) = 0$, then $E'(0) + F(0) = 0$. We are led to the following: When $\Phi'(0) + \Psi(0) = 0$, we get $\langle \vartheta, \theta_0^2 E + \theta_0 F \rangle = \frac{1}{\lambda} \langle w, \theta_0 \Psi + \theta_0^2 \Phi \rangle \neq 0$ from (24a). When $\Phi'(0) + \Psi(0) \neq 0$ and because $\lambda \neq \lambda_{-1}$, then according to (24a) with $c = 0$, we obtain $\langle \vartheta, \theta_0^2 E + \theta_0 F \rangle \neq 0$. Therefore equation (6) is not simplified by $x - c$ for $c \neq 1$. Next, from (23) we have $E'(1) + F(1) = -\Phi(1).

a) If $\Phi(1) \neq 0$, then $E'(1) + F(1) \neq 0$ and the equation (22) cannot be simplified. This means that

$$s_\theta = \max(\deg E - 2, \deg F - 1) = \max(\deg \Phi - 2, \deg \Psi - 1) = s_w + 2.$$

b) If $\Phi(1) = 0$, then $E'(1) + F(1) = 0$ and $\langle \vartheta, \theta_0^2 E + \theta_0 F \rangle = 0$ from (24). Therefore (22) can be simplified by $x - 1$. After simplification, it becomes $\langle \tilde{E}\vartheta \rangle' + \tilde{F}\vartheta = 0$, with $E(x) = x\Phi(x)$ and $F(x) = x(\Psi(x) - (\theta_1 \Phi)(x))$. We have $E'(1) + F(1) = \Psi(1)$. When $\Psi(1) \neq 0$, the above functional equation is not simplified. Consequently, $s_\theta = \max(\deg E - 2, \deg F - 1) = s_w + 1$.

c) If $\Phi(1) = 0$ and $\Psi(1) = 0$, then $E'(1) + \hat{F}(1) = \Psi(1) = 0$. By virtue of (18) and (6) we get $\langle \vartheta, \theta_0^2 E + \theta_1 \hat{F} \rangle = \frac{1}{\lambda} \langle w, \Psi \rangle = 0$. Therefore (34) is simplified by $x - 1$, and $\vartheta$ fulfills $\langle \tilde{E}\vartheta \rangle' + \tilde{F}\vartheta = 0$, where $E(x) = x(\theta_1 \Phi)(x)$ and $\hat{F}(x) = x(\theta_1 \Psi)(x)$. If 1 is a root of $\theta_1 \Phi$, then $\Phi'(1) + \Psi(1) = 0$. Assuming that $E'(1) + \hat{F}(1) = 0$, a simple calculation gives $\langle \vartheta, \theta_0^2 E + \theta_1 \hat{F} \rangle = \frac{1}{\lambda} \langle w, \theta_1 \Psi + \theta_1^2 \Phi \rangle \neq 0$ since $w$ is a semiclassical of class 1 satisfying (9). Hence the functional equation $\langle \tilde{E}\vartheta \rangle' + \tilde{F}\vartheta = 0$ is not simplified and $s_\theta = \max(\deg E - 2, \deg F - 1) = s_w$.

**2 Main Results**

In the sequel we deal with the semiclassical sequence $\{W_n\}_{n \geq 0}$ of class one satisfying (10). Its corresponding regular form $w$ is then semiclassical of class $s_w = 1$ fulfilling the functional equation (6) with $0 \leq \deg \Phi \leq 3$ and $1 \leq \deg \Psi \leq 2$. 
2.1 Characterization of the Polynomials $\Phi$ and $\Psi$

We can usually decompose the polynomials $\Phi$ and $\Psi$ through their odd and even parts. Set

$$
\Phi(x) = \phi(x^2) + x\varphi(x^2), \quad \Psi(x) = \psi(x^2) + x\omega(x^2),
$$

and

$$
(\theta_1, \Phi)(x) = \phi_1(x^2) + x\varphi_1(x^2) \quad \text{and} \quad (\theta_1, \Psi)(x) = \psi_1(x^2) + x\omega_1(x^2).
$$

PROPOSITION 1. Let $w$ be a semiclassical form of class one satisfying (6) and $\{W_n\}_{n \geq 0}$ be its corresponding MOPS fulfilling (10).

a) If $\Phi(1) \neq 0$, then $\phi(x) = \varphi(x) = \frac{1}{2}(x\omega(x) - \psi(x))$.

b) If $\Phi(1) = 0$ and $\Psi(1) \neq 0$, then $\phi(x) = 0$ and $\varphi_1(x) = \omega(x)$.

c) If $\Phi(1) = 0$ and $\Psi(1) = 0$, then $\phi(x) + \varphi(x) = 0$ and $\psi(x) + x\omega(x) = 0$.

PROOF. Set

$$
\tilde{E}(x) = \tilde{E}_c(x^2) + x\tilde{E}_o(x^2); \quad \tilde{F}(x) = \tilde{F}_c(x^2) + x\tilde{F}_o(x^2).
$$

a) $\Phi(1) \neq 0$. According to (37)-(38) and from Theorem 1., we obtain $\tilde{E}_c(x) = x(\phi - \varphi)(x)$, $\tilde{E}_o(x) = x\varphi(x) - \phi(x)$, $\tilde{F}_c(x) = x(\psi - \omega - 2\varphi)(x)$, $\tilde{F}_o(x) = x\omega(x) - \psi(x) - 2\phi(x)$.

b) $\Phi(1) = 0$ and $\Psi(1) \neq 0$. Similar to a), we have $\tilde{E}_c(x) = x\varphi(x)$, $\tilde{E}_o(x) = \phi(x)$, $\tilde{F}_c(x) = x(\omega - \varphi_1)(x)$ and $\tilde{F}_o(x) = (\psi - \phi_1)(x)$. The form $\theta$ is of odd class, then $\tilde{E}_c = \tilde{F}_o = 0$. Hence the conclusion.

c) $\Phi(1) = 0$ and $\Psi(1) = 0$. In this case we have $\tilde{E}_c(x) = x\varphi_1(x)$, $\tilde{E}_o(x) = \phi_1(x)$, $\tilde{F}_c(x) = x\omega_1(x)$, $\tilde{F}_o(x) = \psi_1(x)$. Since $\theta$ is of odd class, $\tilde{F}_c = \tilde{F}_o = 0$. Therefore

$$
\varphi_1 = 0 \quad \text{and} \quad \psi_1 = 0.
$$

Moreover we can write $\Phi(x) = (x + 1)(\theta_1, \Phi)(x) = (x - 1)\phi_1(x^2)$ and $\Psi(x) = (x - 1)x\omega_1(x^2)$. So $\phi = -\phi_1$, $\varphi = -\phi_1$, $\omega = -\omega_1$ and $\psi = x\omega_1$. This gives the desired result.

THEOREM 2. Let $w$ be a semiclassical form of class one satisfying (6) and $\{W_n\}_{n \geq 0}$ be its corresponding (MOPS) fulfilling (10). The functional equation (6) has only one solution given by

$$
\Phi(x) = x^3 - x, \quad \Psi(x) = ax^2 + x + c, \quad a \neq 0, \quad (w)_0 = (w)_1 = 1,
$$

with

$$
a + c + 1 \neq 0; \quad |a + 2| + |a + c + 3| \neq 0 \quad \text{and} \quad |a + 2| + |c - 3| \neq 0.
$$

PROOF. When $\deg \Phi \leq 2$ and $\deg \Psi = 2$, we consider $a \neq 0$, $b$ and $c$ as three complex numbers such that $\Psi(x) = ax^2 + bx + c$. From Proposition 1, we have the following.

i) If $\Phi(1) \neq 0$, then $\phi(x) = \varphi(x)$, and so $\Phi(x) = (x + 1)\phi(x^2)$ from (37). Because $\Phi$ is a monic polynomial of degree at most two, then necessarily $\phi(x) = 1$. In addition, we have $x\omega(x) - \psi(x) = 2$. This implies that $a = b$ and $c = -2$. Thus $\Phi(x) = x + 1$ and $\Psi(x) = ax^2 + ax - 2; a \neq 0$. According to equation (6), we have $\langle w, \Psi(x) \rangle = \langle w, \Phi(x) \rangle = 0$. Therefore $\langle w, \Psi(x) \rangle = 0$ for all $\Psi(x)$. Hence $\Psi(x) = ax^2 + bx + c; a \neq 0$ and $\langle w, \Phi(x) \rangle = 0$. Thus $\Phi(x) = x^2 + bx + c; a \neq 0$. Hence $\Phi(x) = x^2 - x; a \neq 0$. This gives the desired result.
\[
\langle w, x \Psi(x) - \Phi(x) \rangle = 0. \text{ Then } \langle w, ax^2 + ax - 2 \rangle = \langle w, ax^3 + ax^2 - 3x - 1 \rangle = 0. \text{ It is equivalent to }
\]
\[
a(\gamma_1 + 2) - 2 = 0 \quad \text{and} \quad a(\gamma_1 + 1) - 2 = 0,
\]
(41)
since \(\langle w, x \rangle = 1\) and \(\langle w, x^3 \rangle = \langle w, x^2 \rangle = \gamma_1 + 1\). It is easy to see from (41) that \(a = 0\), that is a contradiction with \(\deg \Psi = 2\).

ii) If \(\Phi(1) = 0\) and \(\Psi(1) \neq 0\), then \(\phi(x) = 0\). Therefore \(\Phi(x) = x\), because \(\Phi\) is monic and \(\deg \Phi \leq 2\). This contradicts \(\Phi(1) = 0\).

iii) If \(\Phi(1) = 0\) and \(\Psi(1) = 0\), then \(\Phi(x) = x - 1\) and \(\Psi(x) = a(x^2 - x)\) with \(a \neq 0\). Writing \(\langle w, \Psi(x) \rangle = \langle w, a(x^2 - x) \rangle = 0\), then \(a\gamma_1 = 0\) and so \(\gamma_1 = 0\). It is a contradiction, by virtue of the regularity of the form \(w\).

When \(\deg \Phi = 3\), we obtain \(\deg \phi \leq 1\) and \(\deg \varphi = 1\) from (37). According to Proposition 1, we have the following.

i) If \(\Phi(1) \neq 0\), then \(\phi(x) = \varphi(x)\) and \(\psi(x) = -2\varphi(x) + x\varphi(x)\). We obtain \(\Phi(x) = (x + 1)\varphi(x)\) and \(\Psi(x) = (x^2 + x)\varphi(x) - 2\varphi(x^2)\). Therefore \(\omega\) is a constant polynomial and \(\varphi\) is a monic polynomial of degree one since \(\deg \Phi \leq 2\) and \(\deg \Psi = 3\). Denoting by \(\varphi(x) = x + d\) and \(\omega(x) = e\). We write \(\Phi(x) = (x + 1)(x^2 + d)\) and \(\Psi(x) = (x^2 + x)\varphi(x) - 2\varphi(x^2)\).

Above, we have \(\langle w, \Psi \rangle = \langle w, x \Psi(x) - \Phi(x) \rangle = 0\). It follows \((e - 2)(\gamma_1 + 1) + e - 2d = 0\) and \((e - 2)(\gamma_1 + 1) - 2d = 0\). Hence \(e = 0\) and \(\gamma_1 + d + 1 = 0\). Again, according to equation (6), we have \(\langle (\Phi(x)w)' + \Psi(x)w, x^2 \rangle = 0\), then \(\langle w, x^2(x + d) \rangle = 0\). Since \(x^2 = W_2(x) + \gamma_1 + 1\), we then obtain \(\langle w, (W_2(x) + \gamma_1 + 1)W_2(x) \rangle = 0\). This gives \(\langle w, W_2(x) \rangle = 0\). It is a contradiction with the orthogonality of \(\{W_n\}_{n \geq 0}\).

ii) If \(\Phi(1) = 0\) and \(\Psi(1) = 0\), then \(\phi(x) = -\varphi(x)\) and \(\psi(x) = -x\varphi(x)\). Therefore \(\Psi(x) = (x - x^2)\varphi(x)\), and on account of \(1 \leq \deg \Psi \leq 2\), \(\deg \psi = 0\). Denoting by \(\varphi(x) = a_1\), where \(a_1 \in \mathbb{C} \setminus \{0\}\), since \(\langle w, \Psi \rangle = \langle w, a_1(x - x^2) \rangle = 0\), we have \(a_1\gamma_1 = 0\). It is a contradiction.

iii) If \(\Phi(1) = 0\) and \(\Psi(1) \neq 0\), then \(\phi(x) = 0\) and \(\omega(x) = \varphi_1(x)\). So \(\Phi(x) = x(x^2 - 1)\) and \(\Psi(x) = ax^3 + x + c\). If \(a = 0\), then \(c + 1 = 0\), since \(\langle w, \Psi \rangle = 0\). Thus \(\Psi(x) = x - 1\) which contradicts \(\Psi(1) \neq 0\). Necessarily \(a \neq 0\). Moreover the form \(w\) is of class one, we shall have the condition (9) with \(Z_\Phi = \{-1, 0, 1\}\), which leads to relation (40).

### 2.2 The Computation of \(\gamma_{n+1}\)

We will study the form \(w\) given in Theorem 2. Denoting by \(\alpha = \frac{1}{2}(c - 1)\) and \(\beta = -\frac{1}{2}(a + c + 3)\). The form \(w\) fulfills the following equation

\[
(x(x^2 - 1)w)' + (-2(\alpha + \beta + 2)x^2 + x + 2\alpha + 1)w = 0,
\]
(42)
where

\[
|\alpha + \beta + 1| + |\alpha| \neq 0, \quad \beta + 1 \neq 0, \quad |\alpha + \beta + 1| + |\beta| \neq 0, \quad \alpha + \beta + 2 \neq 0.
\]
(43)

Applying the operator \(\sigma\) in (42) and on account of (2) and (3), we get

\[
((x^2 - x)w)' + (-\alpha + \beta + 2)x + \alpha + 1)u = 0, \quad \langle u \rangle = 1.
\]
(44)
Multiplying (44) by \( x - 1 \), we obtain the functional equation satisfied by the form \( v \)

\[
\left((x^2 - x)v'\right) + (- (\alpha + \beta + 3)x + \alpha + 2)v, \quad (v)_0 = 1.
\]

(45)

Therefore the forms \( u \) and \( v \) are classical. Moreover from a suitable shifting, we obtain

\[
u = \left(\tau_{-\frac{1}{2}} \circ h_{\frac{1}{2}}\right) \mathcal{J}(\alpha, \beta); \quad u = \left(\tau_{\frac{1}{2}} \circ h_{-\frac{1}{2}}\right) \mathcal{J}(\alpha, \beta + 1).
\]

(46)

Where \( \mathcal{J}(\alpha, \beta) \) is the Jacobi form of parameters \( \alpha \) and \( \beta \) satisfying the following functional equation

\[
\left((x^2 - 1)\mathcal{J}(\alpha, \beta)\right)' + (- (\alpha + \beta + 2)x + \alpha - \beta) \mathcal{J}(\alpha, \beta) = 0, \quad (\mathcal{J}(\alpha, \beta))_0 = 1.
\]

It is regular if and only if \( \alpha \neq -n, \quad \beta \neq -n, \quad \alpha + \beta \neq -n, \quad n \geq 1 \). Moreover, the coefficients of its corresponding orthogonal polynomials \( \{P_n^{(\alpha, \beta)}\}_{n \geq 0} \) are given by [1]

\[
\begin{align*}
\beta_n^{(\alpha, \beta)} &= \frac{\alpha^2 - \beta^2}{(2n + \alpha + \beta + 3)(2n + \alpha + \beta + 2)}, \quad n \geq 0, \\
\gamma_{n+1}^{(\alpha, \beta)} &= 4 \frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)}, \quad n \geq 0.
\end{align*}
\]

(47)

PROPOSITION 2. Let \( w \) be the form of class one satisfying (42). The coefficients of its corresponding (MOPS) \( \{W_n\}_{n \geq 0} \) are given by

\[
\gamma_{2n+1} = - \frac{(n+\alpha+\beta+1)(n+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)}, \quad n \geq 0, \\
\gamma_{2n+2} = - \frac{(n+\alpha+\beta+2)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+2)}, \quad n \geq 0.
\]

(48)

PROOF. Let \( \{P_n\}_{n \geq 0} \) be a (MOPS) with respect to the regular form \( u \) and \( \{P^*_n\}_{n \geq 0} \) be the (MOPS) with respect to the regular form \( v \). From (46), we have

\[
P_n(x) = 2^{-n} P_n^{(\alpha, \beta)}(2x - 1), \quad P^*_n(x) = 2^{-n} P_n^{(\alpha, \beta + 1)}(2x - 1), \quad n \geq 0.
\]

(49)

By comparing with (13), (47) and using (8) we get

\[
\begin{align*}
\gamma_{2n+1} \gamma_{2n+2} &= \frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+1)(2n+\alpha+\beta+3)}, \quad n \geq 0, \\
\gamma_{2n+2} \gamma_{2n+3} &= \frac{(n+\alpha+\beta+2)(n+\alpha+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}, \quad n \geq 0.
\end{align*}
\]

(50)

This gives

\[
\frac{\gamma_{2n+3}}{\gamma_{2n+1}} = \frac{(n+\alpha+\beta+2)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)(2n+\alpha+\beta+3)(2n+\alpha+\beta+4)}{(n+\alpha+\beta+1)(n+\beta+1)(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n \geq 0.
\]

By virtue of (50) and from a simple calculation we deduce (48).

REMARK 1. In particular, when \( \alpha = 2^{-1} \) and \( \beta = -2^{-1} \), we obtain the so-called second-order self-associated orthogonal sequence, see [4].
2.3 Integral Representation

Regarding the integral representation of the form $w$ given by (42), we start with the representation of the form $u$. For $\Re(\alpha) > -1$ and $\Re(\beta) > -1$, we have for all $f \in \mathcal{P}$ [1]

$$
\langle u, f \rangle = \left\langle \mathcal{J}(\alpha, \beta), f \left( \frac{x + 1}{2} \right) \right\rangle = \frac{1}{2^{\alpha+\beta+1}} \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{-1}^{1} (1 + x)^{\alpha}(1 - x)^{\beta} f \left( \frac{x + 1}{2} \right) dx.
$$

Using the substitution $t = \frac{x + 1}{2}$, we get

$$
\langle u, f \rangle = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{0}^{1} t^{\alpha}(1 - t)^{\beta} f(t) dt, \ f \in \mathcal{P}. \quad (51)
$$

Next, we decompose the polynomial $f$ as follows:

$$
f(x) = f_1(x^2) + (x - 1)f_2(x^2).
$$

From the fact that $(x - 1)w$ is antisymmetric, we obtain $\langle w, f \rangle = \langle u, f_1 \rangle$. Using again the substitution $t = y^2$ in (51), we obtain

$$
\langle w, f \rangle = 2 \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{0}^{1} y^{2\alpha+1}(1 - y^2)^{\beta} f_1(y^2) dy.
$$

Since for $\Re(\alpha) > -\frac{1}{2}$ and $\Re(\beta) > -1$, $\int_{1}^{1} y \ | \ y \ |^{2\alpha-1} (1 - y^2)^{\beta} f_1(y^2) dy = 0$, the above representation may be written as follows

$$
\langle w, f \rangle = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{-1}^{1} (y^2 + y) \ | \ y \ |^{2\alpha-1} (1 - y^2)^{\beta} f_1(y^2) dy.
$$

Moreover, we have

$$
\int_{-1}^{1} (y^2 + y) \ | \ y \ |^{2\alpha-1} (1 - y^2)^{\beta} (y - 1)f_2(y^2) dy = 0.
$$

Consequently, we get an integral representation of the form $w$ for all $f \in \mathcal{P}$, $\Re\alpha > -\frac{1}{2}$, $\Re\beta > -1$,

$$
\langle w, f \rangle = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{-1}^{1} (y^2 + y) \ | \ y \ |^{2\alpha-1} (1 - y^2)^{\beta} f(y) dy.
$$

References


A Family of Semiclassical Orthogonal Polynomials


