Meromorphic Solutions Of Conjugacy Equations*

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Abstract

This paper characterizes the relation between a conjugacy equation \( \varphi \circ f = g \circ \varphi \) and a permutable functional equation \( \phi \circ f = f \circ \phi \) where \( f : X \to X, g : Y \to Y \) are given self-maps, and \( \varphi, \phi \) are unknown maps. When \( f \) and \( g \) are Möbius transformations, we prove that there exists a bijective meromorphic solution of a conjugacy equation if and only if \( f \) and \( g \) have the same normal form. Moreover, every bijective meromorphic solution is expressed by a permutable meromorphic function with their normal form.

1 Introduction

Let \( X \) and \( Y \) be topological spaces, and let \( f : X \to X \) and \( g : Y \to Y \) be continuous maps. We say that \( f : X \to X \) is topologically conjugate (or simply conjugate) to \( g : Y \to Y \) if there exists a homeomorphism \( \varphi : X \to Y \) satisfying the conjugacy equation (cf. [1,2])

\[
\varphi \circ f = g \circ \varphi,
\]

where \( \circ \) denotes the composition of maps. For instance,

\[
\varphi(z + 1) = \frac{\varphi(z)}{\varphi(z) + 1},
\]

once arose in mathematical competitions or applied mathematics. Taking \( f(z) = z + 1 \) and \( g(z) = z / (z + 1) \), Eq.(2) becomes a conjugacy equation.

In particular, when \( g = f \) and \( \varphi \) is replaced with \( \phi \), the conjugacy equation (1) becomes

\[
\phi \circ f = f \circ \phi,
\]

which is called a permutable functional equation. \( f \) is said to be permutable with \( \phi \) if the relation (3) holds. Permutable functions and close form solutions of functional equations have been extensively studied by many authors (see [3-9]). The monograph [2] collects many results including analytic solutions on a neighborhood of the origin of

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This paper characterizes the relation between Eq.(1) and Eq.(3) for two given \( f : X \to X \) and \( g : Y \to Y \). When \( f \) and \( g \) are Möbius transformations, we prove that there exists a bijective meromorphic solution of a conjugacy equation if and only if \( f \) and \( g \) have the same normal form. Moreover, every bijective meromorphic solution is expressed by a permutable meromorphic function with their normal form. Some examples are illustrated to apply these results.

2 Preliminaries

The following lemma states a relation between permutable functional equation and conjugacy equation.

**Lemma 1.** Let \( \varphi_0 : X \to Y \) be a particular solution of Eq.(1). Then every solution of (1) is given by \( \varphi = \varphi_0 \circ \phi \), where \( \phi : X \to X \) is a solution of Eq.(3).

**Proof.** Since \( \varphi_0 \) is a solution of Eq.(1), we have \( g \circ \varphi_0 = \varphi_0 \circ f \). For any solution \( \phi : X \to X \) of Eq.(3), let \( \varphi = \varphi_0 \circ \phi \), then

\[
\varphi \circ f = \varphi_0 \circ \phi \circ f = \varphi_0 \circ f \circ \phi = g \circ \varphi_0 \circ \phi = g \circ \varphi
\]

This completes the proof.

A Möbius transformation on the complex plane is given by

\[
\ell(z) = \frac{az + b}{cz + d}
\]

where \( a, c, b, d \) are any complex numbers satisfying \( ad - bc \neq 0 \). In case \( c \neq 0 \), this definition is extended to the whole Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) by defining \( \ell(-d/c) = \infty \) and \( \ell(\infty) = a/c \), if \( c = 0 \) we define \( \ell(\infty) = \infty \). This turns \( \ell \) into a bijective meromorphic function from \( \hat{\mathbb{C}} \) to itself.

The set of all Möbius transformations forms a group under composition called the Möbius group. It is the automorphism group of the Riemann sphere, denoted by \( \text{Aut}(\hat{\mathbb{C}}) \).

Let \( GL_2(\mathbb{C}) \) denote the group of all non-singular \( 2 \times 2 \) matrices in the field \( \mathbb{C} \). Define \( h : GL_2(\mathbb{C}) \to \text{Aut}(\hat{\mathbb{C}}) \) by

\[
h\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \frac{az + b}{cz + d}.
\]
The map $h$ is surjective, but not injective because $h(\mu A) = h(A)$ for all nonzero $\mu \in \mathbb{C}$. Define an equivalence in $GL_2(\mathbb{C})$ with $A \sim B$ if and only if $A = \mu B$ and consider the corresponding quotient space $\tilde{GL}_2(\mathbb{C}) := GL_2(\mathbb{C})/\sim$. Then the induced map
\[ \tilde{h} : \tilde{GL}_2(\mathbb{C}) \to \text{Aut}(\tilde{\mathbb{C}}) \]
is bijective.

The following is a well-known fact, which states that the composition of two Möbius transformations corresponds to the multiplication of their corresponding matrices.

**Lemma 2.** Suppose that $A_1, A_2$ are the corresponding matrices of $\ell_1, \ell_2 \in \text{Aut}(\tilde{\mathbb{C}})$, respectively. Then
\[ \ell_1 \circ \ell_2 = h(A_1) \circ h(A_2) = h(A_1A_2). \]

In what follows, we consider Eq.(1) and Eq.(3) where $f, g \in \text{Aut}(\tilde{\mathbb{C}})$.

Using the induced mapping $\tilde{h}$ on the quotient space $\tilde{GL}_2(\mathbb{C})$, the following lemma gives a relation between solutions of the permutable functional equation (3) and a matrix equation
\[ XA = AX, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \]  \hspace{1cm} (6)

**Lemma 3.** Let $A$ be a corresponding matrix of $f \in \text{Aut}(\tilde{\mathbb{C}})$. Then every bijective meromorphic solution of Eq.(3) is given by
\[ \phi(z) = \tilde{h}(X), \]
where $X$ is a solution of Eq.(6).

**Proof.** It is known from the proof of [1, Theorem 11.1.1] that if $\phi$ is a bijective meromorphic function, then $\phi \in \text{Aut}(\tilde{\mathbb{C}})$. So assume $X$ is a corresponding matrix of $\phi$. So we see that
\[ \tilde{h}(X) \circ \tilde{h}(A) = \tilde{h}(A) \circ \tilde{h}(X) \]
It follows from Lemma 2 that
\[ \tilde{h}(XA) = \tilde{h}(AX). \]
Since $\tilde{h}$ is bijective, the matrix equation $XA = AX$ on the quotient space $\tilde{GL}_2(\mathbb{C})$ is equivalent to Eq.(3). Thus every bijective meromorphic solution of Eq.(3) is given by
\[ \phi(z) = \tilde{h}(X), \]
The proof is complete.

If $A \in GL_2(\mathbb{C})$, there exists a nonzero constant $\mu \in \mathbb{C}$ such that $\mu A$ can be transformed into one of the three Jordan canonical forms
\[ J_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J_2 = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \]
where $\lambda \in \mathbb{C}$ is a constant and $\lambda \neq 0, 1$. By Lemmas 2 and 3, it suffices to discuss the case that $A$ can be transformed into one of the above three Jordan canonical forms.
For each $j = 1, 2, 3$ we let $A_j$ denote the collection of matrices $A$ which are similar to $J_j$.

**Lemma 4.** Let $Q$ be an invertible matrix such that $Q^{-1}AQ$ is of a Jordan canonical form. Then Eq.(3) has (i) all Möbius transformations as bijective meromorphic solutions when $A \in A_1$; (ii) infinitely many bijective meromorphic solutions

$$\phi(z) = \hat{h} \left( Q \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} Q^{-1} \right),$$

where $c_1, c_2$ are both arbitrary nonzero complex numbers, when $A \in A_2$; (iii) infinitely many bijective meromorphic solutions

$$\phi(z) = \hat{h} \left( Q \begin{bmatrix} c_1 & c_2 \\ 0 & c_1 \end{bmatrix} Q^{-1} \right),$$

where $c_1, c_2$ are both arbitrary complex numbers and $c_1 \neq 0$, when $A \in A_3$.

**Proof.** Case (i). When $Q^{-1}AQ = J_1$, $f(z) = z$, which commutes with arbitrary functions.

Case (ii). $Q^{-1}AQ = J_2$. All matrices commuting with $J_2$ are of the form

$$C_2 = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix},$$

where $c_1, c_2$ are both arbitrary constants such that $C_2$ is invertible, i.e., $c_1c_2 \neq 0$. It implies that all matrices commuting with $A = QJ_2Q^{-1}$ are of the form $P_2 = QC_2Q^{-1}$. Thus $X = P_2$ is the general solution of Eq.(6). By Lemma 3, the result (8) follows.

Case (iii). $Q^{-1}AQ = J_3$. All matrices commuting with $J_3$ are of the form

$$C_3 = \begin{bmatrix} c_1 & c_2 \\ 0 & c_1 \end{bmatrix},$$

where $c_1, c_2$ are both arbitrary constants such that $C_3$ is invertible, i.e., $c_1 \neq 0$. It implies that all matrices commuting with $A = QJ_3Q^{-1}$ are of the form $P_3 = QC_3Q^{-1}$. Thus $X = P_3$ is the general solution of Eq.(6). By Lemma 3, the result (9) follows.

We give an example to illustrate the use of the formulae obtained above.

**Example 1.** Consider $f(z) = \frac{7z - 3}{18z - 8}$, which corresponds to

$$A = \begin{bmatrix} 7 & -3 \\ 18 & -8 \end{bmatrix}.$$

Choosing

$$Q = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

we have

$$Q^{-1}AQ = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$
Then $A \in A_2$. From (8), we see that Eq.(3) has infinitely many bijective meromorphic solutions

$$
\phi(z) = \tilde{h}\left(Q\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}Q^{-1}\right) = \tilde{h}\left(\begin{bmatrix} -2c_1 + 3c_2 & c_1 - c_2 \\ -6c_1 + 6c_2 & 3c_1 - 2c_2 \end{bmatrix}\right) = (-2c_1 + 3c_2)z + c_1 - c_2,
$$

$$
= (\frac{-2 + 3\mu}{-6 + 6\mu})z + 1 - \mu
$$

$$
= (\frac{-2 + 3\mu}{-6 + 6\mu})z + 3 - 2\mu,
$$

where $\mu := c_2/c_1$ and $\mu \in \mathbb{C}$ is an arbitrary nonzero constant.

We consider normal forms of the Möbius group under $\text{Aut}(\tilde{\mathbb{C}})$-conjugacy.

**Lemma 5.** Under $\text{Aut}(\tilde{\mathbb{C}})$-conjugacy, the Möbius group has only three normal forms:

1. $e_1(z) = z$;
2. $e_2(z) = \lambda z$, $\lambda \neq 0, 1$;
3. $e_3(z) = z + 1$.

**Proof.** Suppose that $\ell \in \text{Aut}(\tilde{\mathbb{C}})$ corresponds to a matrix $A$ which can be transformed into one of the three Jordan canonical forms in (7). So assume that the Jordan canonical form of $A$ is $J_i$ for some $i$. Then there exists a nonsingular $2 \times 2$ matrix $Q$ such that $J_i = Q^{-1}AQ$. By Lemma 2, we have

$$
e_i(z) = \tilde{h}(J_i) = \tilde{h}(Q^{-1}AQ) = \tilde{h}(Q^{-1}) \circ \tilde{h}(A) \circ \tilde{h}(Q) = \tilde{h}^{-1}(Q) \circ \tilde{h}(A) \circ \tilde{h}(Q) = \tilde{h}^{-1}(Q) \circ \ell(z) \circ \tilde{h}(Q).
$$

Since $\tilde{h}^{-1}(Q), \tilde{h}(Q) \in \text{Aut}(\tilde{\mathbb{C}})$, $\ell(z)$ is conjugate to $e_i(z)$ under $\text{Aut}(\tilde{\mathbb{C}})$-conjugacy.

Obviously the three normal forms above are not conjugate to each other under $\text{Aut}(\tilde{\mathbb{C}})$-conjugacy. This completes the proof.

### 3 Conjugacy Equation

We have the following main result.

**Theorem 1.** Suppose that $f, g \in \text{Aut}(\tilde{\mathbb{C}})$. Then there exists a bijective meromorphic solution of Eq.(1) if and only if $f$ and $g$ have the same normal form. Moreover, suppose $\varphi_j \in \text{Aut}(\tilde{\mathbb{C}})$, $j = 1, 2$ satisfy

$$
\varphi_1^{-1} \circ f \circ \varphi_1 = \varphi_2^{-1} \circ g \circ \varphi_2 = e_i \quad \text{for some } i.
$$

Then every bijective meromorphic solutions of Eq.(1) is given by

$$
\varphi = \varphi_2 \circ \phi \circ \varphi_1^{-1},
$$

where $\phi$ is a bijective meromorphic solution of the equation $\phi \circ e_i = e_i \circ \phi$. 

PROOF. By (10), we have $\varphi^{-1}_1 \circ f = e_i \circ \varphi^{-1}_1$ and $g \circ \varphi_2 = \varphi_2 \circ e_i$. For any bijective meromorphic solution $\phi$ of the equation $\phi \circ e_i = e_i \circ \phi$, let $\varphi = \varphi_2 \circ \phi \circ \varphi^{-1}_1$.

Then

$$
\varphi \circ f = \varphi_2 \circ \phi \circ \varphi^{-1}_1 \circ f = \varphi_2 \circ \phi \circ \varphi^{-1}_1 = \varphi_2 \circ e_i \circ \phi \circ \varphi^{-1}_1 = g \circ \varphi_2 \circ \phi \circ \varphi^{-1}_1 = g \circ \varphi.
$$

Conversely, if there exists a bijective meromorphic solution $\varphi$ of Eq.(1), then $f = \varphi^{-1} \circ g \circ \varphi$. Therefore $f$ and $g$ have the same normal form. This completes the proof.

EXAMPLE 2. Consider Eq.(2). Choosing $\varphi_0(z) = 1/z$, we have $\varphi_0 \circ f = g \circ \varphi_0$. By Lemma 1, it suffices to solve $\phi \circ f = f \circ \phi$. In fact, the general solution of $\phi(z + 1) = \phi(z) + 1$ is given by $\phi(z) = \Theta(z) + z$, where $\Theta(z) = \Theta(z + 1)$ is an arbitrary periodic function with unit period. By Lemma 1, the general solution of Eq.(2) is given by

$$
\varphi(z) = \varphi_0 \circ \phi(z) = \frac{1}{\Theta(z) + z}.
$$

Remark that Eq.(2) was discussed in [2, pp.390-391, Theorem 10.1.2]. Their result shows that the only convex or concave solutions of Eq.(2) are $\varphi = 0$ and $\varphi(x) = 1/(x + d)$, where $d \in \mathbb{R}$ is an arbitrary constant. Clearly they are two particular solutions.

EXAMPLE 3. Consider the functional equation

$$
\varphi \left( \frac{31z - 12}{70z - 27} \right) = \frac{43\varphi(z) - 24}{70\varphi(z) - 39}. \tag{11}
$$

Put

$$
f(z) = \frac{31z - 12}{70z - 27}, \quad g(z) = \frac{43z - 24}{70z - 39}.
$$

By Lemma 5, choose

$$
\varphi_1(z) = \frac{3z + 2}{7z + 5}, \quad \varphi_2(z) = \frac{3z + 4}{5z + 7}.
$$

Then $\varphi^{-1}_1 \circ f \circ \varphi_1(z) = \varphi^{-1}_2 \circ g \circ \varphi_2(z) = 3z$. From Lemma 4, all bijective meromorphic solutions of $\phi(3z) = 3\phi(z)$ are given by $\phi(z) = \mu z$, where $\mu \in \mathbb{C}$ is an arbitrary nonzero constant. By Theorem 1, all bijective meromorphic solutions of Eq.(11) are given by

$$
\varphi(z) = \varphi_2 \circ \phi \circ \varphi^{-1}_1(z) = \frac{(15\mu - 28)z - 6\mu + 12}{(25\mu - 49)z - 10\mu + 21}.
$$

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References


