Dichotomy Of Poincare Maps And Boundedness Of Some Cauchy Sequences*

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Abstract

Let \( f(U(p, q)) \) be the \( N \)-periodic discrete evolution family of \( m \times m \) matrices having complex scalers as entries generated by \( L(C^m) \)-valued, \( N \)-periodic sequence of \( m \times m \) matrices \( (A_n) \) where \( N \geq 2 \) is a natural number. We proved that the Poincare map \( U(N, 0) \) is dichotomic if and only if the matrix \( V_\mu = \sum_{\nu=1}^{N} U(N, \nu) e^{i\mu\nu} \) is invertible and there exists a projection \( P \) which commutes with the map \( U(N, 0) \) and the matrix \( V_\mu \), such that for each \( \mu \in \mathbb{R} \) and each vector \( b \in C^m \) the solutions of the discrete Cauchy sequences \( x_{n+1} = A_n x_n + e^{i\mu n} Pb, \quad x_0 = 0 \) and \( y_{n+1} = A_n^{-1} y_n + e^{i\mu n} (I - P)b, \quad y_0 = 0 \) are bounded.

1 Introduction

It is well-known, see [2], that a matrix \( A \) is dichotomic, i.e. its spectrum does not intersect the unit circle if and only if there exists a projector, i.e. an \( m \times m \) matrix \( P \) satisfying \( P^2 = P \), which commutes with \( A \) and has the property that for each real number \( \mu \) and each vector \( b \in C^m \), the following two discrete Cauchy problems

\[
\begin{cases}
  x_{n+1} = A x_n + e^{i\mu n} Pb, & n \in \mathbb{Z}_+ \\
  x_0 = 0
\end{cases}
\]

and

\[
\begin{cases}
  y_{n+1} = A^{-1} y_n + e^{i\mu n} (I - P)b, & n \in \mathbb{Z}_+ \\
  y_0 = 0
\end{cases}
\]

have bounded solutions. In particular, the spectrum of \( A \) belongs to the interior of the unit circle if and only if for each real number \( \mu \) and each \( m \)-vector \( b \), the solution of the Cauchy problem (1) is bounded. Continuous version of the above result is given in [4].

On the other hand, in [3], it is shown that an \( N \)-periodic evolution family \( U = \{U(p, q)\} \) of bounded linear operators acting on a complex space \( X \), is uniformly

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exponentially stable, i.e. the spectral radius of the Poincare map \( U(N,0) \) is less than one, if and only if for each real number \( \mu \) and each \( N \)-periodic sequence \((z_n)\) decaying to \( n = 0 \), we have

\[
\sup_{n \geq 1} \left\| \sum_{k=1}^{n} e^{ink} U(n,k) z_{k-1} \right\| = M(\mu, b) < \infty.
\]

Recently in [1], it is proved that the spectral radius of the matrix \( U(N,0) \) is less than one, if for each real \( \mu \) and each \( m \)-vector \( b \), the operator \( V_\mu := \sum_{\nu=1}^{N} e^{i\mu \nu} U(N,\nu) \) is invertible and

\[
\sup_{n \geq 1} \left\| \sum_{j=1}^{kN} e^{i\mu(j-1)} U(kN,j) b \right\| < \infty.
\]

This note is a continuation of the latter quoted paper. In fact, we prove that the matrix \( U(N,0) \) is dichotomic if and only if for each real \( \mu \) and each \( m \)-vector \( b \), the operator \( V_\mu := \sum_{\nu=1}^{N} e^{i\mu \nu} U(N,\nu) \) is invertible and solutions of the two discrete Cauchy sequences like \((A, P \cdot b, x_0, 0)\) are bounded.

2 Preliminary Results

Consider the following Cauchy Problem

\[
\begin{align*}
  z_{n+1} &= Az_n, & z_n \in \mathbb{C}^m, & n \in \mathbb{Z}_+ \\
  z_n(0) &= z_0.
\end{align*}
\]

where \( A \) is an \( m \times m \) matrix. It is easy to check that the solution of (3) is \( A^n z_0 \).

Consider the following lemma which is used in Theorem 1.

**LEMMA 1.** Let \( N \geq 1 \) be a natural number. If \( q_n \) is a polynomial of degree \( n \) and \( \Delta^N q_n = 0 \) for all \( n = 0,1,2 \ldots \) where \( \Delta z_n = z_{n+1} - z_n \) then \( q \) is a \( \mathbb{C}^m \)-valued polynomial of degree less than or equal to \( N - 1 \).

For proof see [2].

Let \( p_A \) be the characteristic polynomial associated with the matrix \( A \) and let \( \sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \), \( k \leq m \) be its spectrum. There exist integer numbers \( m_1, m_2, \ldots, m_k \geq 1 \) such that

\[
p_A(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}, \quad m_1 + m_2 + \cdots + m_k = m.
\]

Then in [2] we have the following theorem.

**THEOREM 1.** For each \( z \in \mathbb{C}^m \) there exists \( w_j \in W_j := \ker(A - \lambda_j I)^{m_j} \), \( j \in \{1,2,\ldots,k\} \) such that

\[
A^n z = A^n w_1 + A^n w_2 + \cdots + A^n w_k.
\]

Moreover, if \( w_j(n) := A^n w_j \) then \( w_j(n) \in W_j \) for all \( n \in \mathbb{Z}_+ \) and there exist a \( \mathbb{C}^m \)-valued polynomials \( q_j(n) \) with \( \deg(q_j) \leq m_j - 1 \) such that

\[
w_j(n) = \lambda_j^n q_j(n), \quad n \in \mathbb{Z}_+, \quad j \in \{1,2,\ldots,k\}.
\]
FROOF. Indeed from the Cayley-Hamilton theorem and using the well known fact that
\[
\ker[pq(A)] = \ker[p(A)] \oplus \ker[q(A)]
\]
whenever the complex valued polynomials \( p \) and \( q \) are relatively prime, it follows that
\[
\mathbb{C}^m = W_1 \oplus W_2 \oplus \cdots \oplus W_k.
\]  
(4)

Let \( z \in \mathbb{C}^m \). For each \( j \in \{1, 2, \ldots, k\} \) there exists a unique \( w_j \in W_j \) such that
\[
z = w_1 + w_2 + \cdots + w_k
\]
and then
\[
A^n z = A^n w_1 + A^n w_2 + \cdots + A^n w_k, \quad n \in \mathbb{Z}_+.
\]

Let \( q_j(n) = \lambda_j^{-n} w_j(n) \). Successively one has
\[
\Delta q_j(n) = \Delta(\lambda_j^{-n} w_j(n)) \\
= \Delta(\lambda_j^{-n} A^n w_j) \\
= \lambda_j^{-(n+1)} A^{n+1} w_j - \lambda_j^{-n} A^n w_j \\
= \lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j.
\]

Again taking \( \Delta \),
\[
\Delta^2 q_j(n) = \Delta[\Delta q_j(n)] \\
= \Delta[\lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j] \\
= \lambda_j^{-(n+2)} (A - \lambda_j I) A^{n+1} w_j - \lambda_j^{-(n+1)} (A - \lambda_j I) A^n w_j \\
= \lambda_j^{-(n+2)} (A - \lambda_j I)^2 A^n w_j.
\]

Continuing up to \( m_j \) we get \( \Delta^{m_j} q_j(n) = \lambda_j^{-(n+m_j)} (A - \lambda_j I)^{m_j} A^n w_j \). But \( w_j(n) \) belongs to \( W_j \) for each \( n \in \mathbb{Z}_+ \). Thus \( \Delta^{m_j} q_j(n) = 0 \). Using Lemma 1, we can say that the degree of polynomial \( q_j(n) \) is less than or equal to \( m_j - 1 \).

3 Dichotomy and Boundedness

A family \( \mathcal{U} = \{U(p, q) : (p, q) \in \mathbb{Z}_+ \times \mathbb{Z}_+\} \) of an \( m \times m \) complex valued matrices is called discrete periodic evolution family if it satisfies the following properties.

1. \( U(p, q)U(q, r) = U(p, r) \) for all \( p \geq q \geq r \geq 0 \);
2. \( U(p, p) = I \) for all \( p \geq 0 \) and
3. there exists a fixed \( N \geq 2 \) such that \( U(p + N, q + N) = U(p, q) \) for all \( p, q \in \mathbb{Z}_+, \quad p \geq q \).
Let us consider the following discrete Cauchy problem:

\[
\begin{cases}
    z_{n+1} = A_n z_n + e^{i\mu n} b, & n \in \mathbb{Z}_+ \\
    z_0 = 0,
\end{cases}
\]

where the sequence \((A_n)\) is \(N\)-periodic, i.e. \(A_{n+N} = A_n\) for all \(n \in \mathbb{Z}_+\) and a fixed \(N \geq 2\). Let

\[ U(n, j) = \begin{cases} 
    A_{n-1} A_{n-2} \cdots A_j & \text{if } j \leq n - 1, \\
    I & \text{if } j = n,
\end{cases} \]

then, the family \(\{U(n, j)\}_{n \geq j \geq 0}\) is a discrete \(N\)-periodic evolution family and the solution \((z_n)\) of the Cauchy problem \((A_n, \mu, b)_0\) is given by:

\[ z_n = \sum_{j=1}^{n} U(n, j) e^{i\mu (j-1)} b \]

Let us denote by \(C_1 = \{z \in \mathbb{C} : |z| = 1\}\), \(C_1^+ = \{z \in \mathbb{C} : |z| > 1\}\) and \(C_1^- = \{z \in \mathbb{C} : |z| < 1\}\). Clearly \(\mathbb{C} = C_1 \cup C_1^+ \cup C_1^-\). Then with the help of above partition of \(\mathbb{C}\) for matrix \(A\) we give the following definition:

**DEFINITION 1.** The matrix \(A\) is called:

(i) stable if \(\sigma(A)\) is the subset of \(C_1^-\) or, equivalently, if there exist two positive constants \(N\) and \(\nu\) such that \(\|A^n\| \leq Ne^{-\nu n}\) for all \(n = 0, 1, 2, \ldots\),

(ii) expansive if \(\sigma(A)\) is the subset of \(C_1^+\) and

(iii) dichotomic if \(\sigma(A)\) have empty intersection with set \(C_1\).

It is clear that any expansive matrix \(A\) whose spectrum consists of \(\lambda_1, \lambda_2, \ldots, \lambda_k\) is an invertible one and its inverse is stable, because

\[ \sigma(A^{-1}) = \left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_k} \right\} \subset C_1^- . \]

Let \(L := U(N, 0), V_\mu = \sum_{\nu=1}^{N} U(N, \nu) e^{i\mu \nu}\) and \(A_i A_j = A_j A_i\) for any \(i, j \in \{1, 2, \ldots, n\}\).

We recall that a linear map \(P\) acting on \(\mathbb{C}^m\) is called projection if \(P^2 = P\).

**THEOREM 2.** Let \(N \geq 2\) be a fixed integer number. The matrix \(L\) is dichotomic if and only if the matrix \(V_\mu\) is invertible and there exists a projection \(P\) having the property \(PL = LP\) and \(PV_\mu = V_\mu P\) such that for each \(\mu \in \mathbb{R}\) and each vector \(b \in \mathbb{C}^m\) the solutions of the following discrete Cauchy problems

\[
\begin{cases}
    x_{n+1} = A_n x_n + e^{i\mu n} P b, & n \in \mathbb{Z}_+ \\
    x_0 = 0
\end{cases}
\]

and

\[
\begin{cases}
    y_{n+1} = A_n^{-1} y_n + e^{i\mu n} (I - P) b, & n \in \mathbb{Z}_+ \\
    y_0 = 0
\end{cases}
\]
are bounded.

**PROOF. Necessity:** Working under the assumption that $L$ is a dichotomic matrix we may suppose that there exists $\eta \in \{1, 2, \ldots, \xi\}$ such that

$$|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_\eta| < 1 < |\lambda_{\eta+1}| \leq \cdots \leq |\lambda_\xi|.$$ 

Having in mind the decomposition of $\mathbb{C}^m$ given by (4) let us consider

$$X_1 = W_1 \oplus W_2 \oplus \cdots \oplus W_\eta, \quad X_2 = W_{\eta+1} \oplus W_{\eta+2} \oplus \cdots \oplus W_\xi.$$

Then $\mathbb{C}^m = X_1 \oplus X_2$. Define $P : \mathbb{C}^m \to \mathbb{C}^m$ by $Px = x_1$, where $x = x_1 + x_2$, $x_1 \in X_1$ and $x_2 \in X_2$. It is clear that $P$ is a projection. Moreover for all $x \in \mathbb{C}^m$ and all $n \in \mathbb{Z}_+$, this yields

$$PL^k x = P(L^k(x_1 + x_2)) = P(L^k(x_1) + L^k(x_2)) = L^k(x_1) = L^k P x,$$

where the fact that $X_1$ is an $L^k-$ invariant subspace, was used. Then $PL^k = L^k P$.

Similarly by using the fact that $X_1$ and $X_2$ are $V_\mu$ invariant subspaces we can prove that $PV_\mu = V_\mu P$. We know that the solution of the Cauchy problem (5) is:

$$x_n = \sum_{j=1}^{n} U(n, j) e^{j(i-1)} P b.$$ 

Put $n = Nk + r$, where $r = 0, 1, 2, \ldots, N - 1$. Then

$$x_{Nk+r} = \sum_{j=1}^{Nk+r} U(Nk+r, j) e^{j(i-1)} P b.$$ 

Let

$$A_\nu = \{\nu, \nu + N, \ldots, \nu + (k-1)N\}, \quad \text{where } \nu \in \{1, 2, \ldots, N\}$$

and

$$R = \{kN + 1, kN + 2, \ldots, kN + r\}.$$

Then

$$R \cup (\cup_{\nu=1}^{N} A_\nu) = \{1, 2, \ldots, n\}.$$

Thus

$$x_{Nk+r} = e^{-i\mu} \sum_{\nu=1}^{N} \sum_{j \in A_\nu} U(Nk+r, j) e^{i\mu j} P b + e^{-i\mu} \sum_{j \in R} U(Nk+r, j) e^{i\mu j} P b$$

$$= e^{-i\mu} \sum_{\nu=1}^{N} \sum_{s=0}^{k-1} U(Nk+r, \nu + sN) e^{i\mu (\nu + sN)} P b +$$

$$e^{-i\mu} \sum_{\rho=1}^{r} U(Nk+r, Nk+\rho) e^{i\mu (kN+\rho)} P b.$$
\[ Zada \text{ et al.} \]

\[
e^{-i\mu} \sum_{\nu=1}^{N} \sum_{s=0}^{k-1} U(r, 0) U(N, 0) (k-s-1) U(N, \nu) e^{i\mu(\nu+sN)} P b + \\
e^{-i\mu} \sum_{\rho=1}^{r} U(r, \rho) e^{i\mu(kN+\rho)} P b.
\]

Let \( z_\mu = e^{i\mu N} \), also we know that \( L = U(N, 0) \), thus

\[
x_{Nk+r} = e^{-i\mu} U(r, 0) \sum_{s=0}^{k-1} L^{k-s-1} z_\mu^s \sum_{\nu=1}^{N} U(N, \nu) e^{i\mu \nu} P b + \\
e^{-i\mu} z_\mu^k \sum_{\rho=1}^{r} U(r, \rho) e^{i\mu \rho} P b
\]

\[ = e^{-i\mu} U(r, 0) \left( L^{k-1} z_\mu^0 + L^{k-2} z_\mu^1 + \cdots + L^0 z_\mu^{k-1} \right) \sum_{\nu=1}^{N} U(N, \nu) e^{i\mu \nu} P b + \\
e^{-i\mu} z_\mu^k \sum_{\rho=1}^{r} U(r, \rho) e^{i\mu \rho} P b.
\]

We know that \( \sum_{\nu=1}^{N} U(N, \nu) e^{i\mu \nu} = V_\mu \) thus

\[
x_{Nk+r} = e^{-i\mu} U(r, 0) \left( L^{k-1} z_\mu^0 + L^{k-2} z_\mu^1 + \cdots + L^0 z_\mu^{k-1} \right) V_\mu P b + \\
e^{-i\mu} z_\mu^k \sum_{\rho=1}^{r} U(r, \rho) e^{i\mu \rho} P b.
\]

By our assumption we know that \( L \) is dichotomic and \( |z_\mu| = 1 \) thus \( z_\mu \) is contained in the resolvent set of \( L \) therefore the matrix \( (z_\mu I - L) \) is an invertible matrix. Thus

\[
x_{Nk+r} = e^{-i\mu} U(r, 0) (z_\mu I - L)^{-1} (z_\mu^k I - L^k) V_\mu P b + e^{-i\mu} z_\mu^k \sum_{\rho=1}^{r} U(r, \rho) e^{i\mu \rho} P b
\]

\[ = e^{-i\mu} U(r, 0) (z_\mu I - L)^{-1} (z_\mu^k I - L^k) V_\mu b + e^{-i\mu} z_\mu^k \sum_{\rho=1}^{r} U(r, \rho) e^{i\mu \rho} P b.
\]

We know that \( V_\mu \) is a surjective map, so there exists \( b' \) such that \( V_\mu b = b' \) then

\[
x_{Nk+r} = e^{-i\mu} U(r, 0) (z_\mu I - L)^{-1} (z_\mu^k I - L^k) P b' + e^{-i\mu} z_\mu^k \sum_{\rho=1}^{r} U(r, \rho) e^{i\mu \rho} P b.
\]

Taking norm of both sides

\[
\| x_{Nk+r} \| = \| e^{-i\mu} U(r, 0) (z_\mu I - L)^{-1} (z_\mu^k I - L^k) P b' + e^{-i\mu} z_\mu^k \sum_{\rho=1}^{r} U(r, \rho) e^{i\mu \rho} P b \|
\]
It is easy to check that

\[ k \text{ as above we obtained that} \]

\[ A \]

Thus, using the above, we obtain that

\[ 2 f \]

\[ i \]

\[ N_k \]

\[ 2 \]

\[ \|

Using THEOREM 1, we have

\[ L^k b' = \lambda_1^k p_1(k) + \lambda_2^k p_2(k) + \cdots + \lambda_k^k p_k(k), \]

Thus

\[ PL^k b' = \lambda_1^k p_1(k) + \lambda_2^k p_2(k) + \cdots + \lambda_k^k p_k(k), \]

where each \( p_i(k) \) are \( \mathbb{C}^{m_i} \)-valued polynomials with degree at most \( (m_i - 1) \) for any \( i \in \{1, 2, \ldots, \xi\} \). From hypothesis we know that \( |\lambda_i| < 1 \) for each \( i \in \{1, 2, \ldots, \eta\} \). Thus \( ||PL^k b'|| \to 0 \) when \( k \to \infty \) and so \( x_{Nk+r} \) is bounded for any \( r = 0, 1, 2, \ldots, N - 1 \). Thus \( x_n \) is bounded. For the second Cauchy problem: We have

\[ y_n = \sum_{j=1}^{n} U^{-1}(n, j) e^{i\mu(j-1)} (I - P) b. \]

where

\[ U^{-1}(n, j) = \begin{cases} A_{n-1}^{-1} A_{n-2}^{-1} \cdots A_j^{-1} & \text{if } j \leq n - 1, \\ I & \text{if } j = n. \end{cases} \]

It is easy to check that \( U^{-1}(n, j) \) is also a discrete evaluation family. By putting \( n = Nk + r \), where \( r = 0, 1, 2, \ldots, N - 1 \). Then

\[ y_{Nk+r} = \sum_{j=1}^{Nk+r} U^{-1}(Nk+r, j) e^{i\mu(j-1)} (I - P) b. \]

As \( A_i A_j = A_j A_i \) for all \( i, j \in \{1, 2, \ldots, n\} \) thus \( L^{-1} = U^{-1}(N, 0) \). By similar procedure as above we obtained that

\[ ||y_{Nk+r}|| = \|U^{-1}(r, 0)||((z_1 I - L)^{-1})^{-1}||(I - P) V_\mu(b)|| + \|U^{-1}(r, 0)||((z_1 I - L)^{-1})^{-1}||L^{-k}(I - P) V_\mu(b)|| + \sum_{\rho=1}^{r} \|U^{-1}(r, \rho)(I - P) b\|. \]

Since \( (I - P) V_\mu b \in X_2 \) the assertion would follow. But

\[ X_2 = W_{\eta+1} \oplus W_{\eta+2} \oplus \cdots \oplus W_\xi. \]

Each vector from \( X_2 \) can be represented as a sum of \( \xi - \eta \) vectors \( w_{\eta+1}, w_{\eta+2}, \ldots, w_\xi \). It would be sufficient to prove that \( L^{-k} w_j \to 0 \), for any \( j \in \{\eta + 1, \ldots, \xi\} \). Let \( W \in \)
\{W_{\gamma+1}, W_{\gamma+2}, \ldots, W_{\xi}\}, say W = \ker(L - \lambda I)^\gamma, where \gamma \geq 1 is an integer number and |\lambda| > 1. Consider \(r_1 \in W \setminus \{0\}\) such that \((L - \lambda I)r_1 = 0\) and let \(r_2, r_3, \ldots, r_\gamma\) given by \((L - \lambda I)r_j = r_{j-1}, j = 2, 3, \ldots, \gamma\). Then \(B := \{r_1, r_2, \ldots, r_\gamma\}\) is a basis in \(Y\). It is then sufficient to prove that \(L^{-k}r_j \to 0\) for any \(j = 1, 2, \ldots, \gamma\). For \(j = 1\) we have that \(L^{-k}r_1 = \frac{1}{\lambda^k}r_1 \to 0\). For \(j = 2, 3, \ldots, \gamma\), denote \(X_k = L^{-k}r_j\). Then \((L - \lambda I)^\gamma X_k = 0\) i.e.

\[
X_k - C_\gamma X_{k-1} \alpha + C_\gamma^2 X_{k-2} \alpha^2 + \cdots + C_\gamma^\gamma X_{k-\gamma} \alpha^\gamma = 0, \quad \text{for all } k \geq \gamma
\]  

(7)

where \(\alpha = \frac{1}{\lambda}\). Passing for instance at the components, it follows that there exists a \(\mathbb{C}^m\)-valued polynomial \(P_\gamma\) having degree at most \(\gamma - 1\) and verifying (7) such that \(X_k = \alpha^k P_\gamma(k)\). Thus \(X_k \to 0\) as \(k \to \infty\), i.e. \(L^{-k}r_j \to 0\) for any \(j \in \{1, 2, \ldots, \gamma\}\). Thus \((y_\mu)\) is bounded.

**Sufficiency:** Suppose to the contrary that the matrix \(L\) is not dichotomic. Then \(\sigma(L) \cap \Gamma_1 \neq \emptyset\). Let \(\omega \in \sigma(L) \cap \Gamma_1\). Then there exists a nonzero \(y \in \mathbb{C}^m\) such that \(Ly = \omega y\). It is easy to see that \(L^k y = \omega^k y\). Choose \(\mu_0 \in \mathbb{R}\) such that \(e^{i\mu_0 N} = \omega\). We know that

\[
x_{Nk+r}(\mu_0, b) = e^{-i\mu_0} U(r, 0) \left( L^{k-1} z_{\mu_0}^0 + L^{k-2} z_{\mu_0}^1 + \cdots + L^{k-1} z_{\mu_0}^0 \right) P V_{\mu_0} b +
\]

\[
e^{-i\mu_0} e^{i\mu_0} \sum_{\rho=1}^r U(r, \rho) e^{i\mu_0 \rho} P b.
\]

But \(V_{\mu_0}\) is surjective, thus there exists \(b_0 \in \mathbb{C}^m\) such that \(V_{\mu_0} b_0 = y\), so

\[
x_{Nk+r}(\mu_0, b_0) = e^{-i\mu_0} U(r, 0) \left( L^{k-1} z_{\mu_0}^0 + L^{k-2} z_{\mu_0}^1 + \cdots + L^{k-1} z_{\mu_0}^0 \right) P y +
\]

\[
e^{-i\mu_0} e^{i\mu_0} \sum_{\rho=1}^r U(r, \rho) e^{i\mu_0 \rho} P b_0
\]

\[
= e^{-i\mu_0} U(r, 0) \left( P L^{k-1} y z_{\mu_0}^0 + P L^{k-2} y z_{\mu_0}^1 + \cdots + P L^{k-1} y z_{\mu_0}^0 \right) +
\]

\[
e^{-i\mu_0} e^{i\mu_0} \sum_{\rho=1}^r U(r, \rho) e^{i\mu_0 \rho} P b
\]

\[
= e^{-i\mu_0} U(r, 0) P [k e^{-i\mu_0} z_{\mu_0}^{k-1}] + e^{-i\mu_0} e^{i\mu_0} \sum_{\rho=1}^r U(r, \rho) e^{i\mu_0 \rho} P b
\]

Clearly

\[
x_{nk}(\mu_0, b_0) \to \infty \text{ when } k \to \infty.
\]

Thus a contradiction arises. In [1] an example, in terms of stability is given which shows that the assumption on invertibility of \(V_{\mu}\), for each real number \(\mu\), cannot be removed.
References


