On A Metaharmonic Boundary Value Problem

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Received 1 November 2010

Abstract

In this paper we develop maximum principles for solutions of metaharmonic equations defined on arbitrary \( n \) dimensional domains. As a consequence we obtain an uniqueness result for the corresponding metaharmonic boundary value problem.

1 Introduction

In the paper [4] we showed that if \( a_1, a_3 \geq 0 \) (\( a_1, a_3 \) constants), \( a_2(x) \geq 0 \), \( a_4(x) > 0 \) in \( \Omega \subset \mathbb{R}^2 \) and the curvature of \( \partial \Omega \in C^{2+\varepsilon} \) is strictly positive, then the boundary value problem

\[
\begin{aligned}
\Delta^4 u - a_1 \Delta^3 u + a_2(x) \Delta^2 u - a_3 \Delta u + a_4(x) u &= f \quad \text{in } \Omega, \\
u &= g, \Delta u = h, \Delta^2 u = i, \Delta^3 u = j \quad \text{on } \Omega
\end{aligned}
\]

has at most a classical solution in \( C^8(\Omega) \cap C^6(\overline{\Omega}) \).

Using a generalized maximum principle we are able here to extend the above mentioned result for a the \( m \) metaharmonic problem

\[
\begin{aligned}
\Delta^m u - a_{m-1}(x) \Delta^{m-1} u + a_{m-2}(x) \Delta^{m-2} u + \cdots + (-1)^m a_0(x) u &= f \quad \text{in } \Omega, \\
u &= g_1, \Delta u = g_2, \ldots, \Delta^{m-1} u = g_m \quad \text{on } \Omega
\end{aligned}
\]

where \( a_i, i = 0, \ldots, m - 1 \), are bounded in the bounded domain \( \Omega \subset \mathbb{R}^n, n \geq 2 \). Here we deal with classical solutions \( u \) of (2), i.e., \( u \in C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega}), m \geq 3 \).

This result generalizes the result of Dunninger [5] (the case \( m = 2, n \geq 2, a_1 = 0, a_0 \equiv \text{constant} \geq 0 \) and \( \Omega \) arbitrary), Schaefer [7] (the case curvature of \( \partial \Omega > 0, m = n = 2 \)), Schaefer [8] (the case \( a_2, a_1 \geq 0, a_0 > 0 \) with \( m = 3, n = 2 \), curvature of \( \partial \Omega > 0 \)), S. Goyal and V. Goyal [6] and Danet [3] (the variable coefficient case with \( m = 3 \) and \( \Omega \subset \mathbb{R}^n \) arbitrary).

Throughout this paper we shall assume that \( \Omega \subset \mathbb{R}^n, n \geq 2 \) is a bounded domain, \( m \geq 3 \) and the coefficients \( a_i, i = 0, \ldots, m - 1 \) are bounded in \( \Omega \). Also we shall suppose that \( a_0 \neq 0 \). \text{diam} \( \Omega \) will denote the diameter of \( \Omega \).
2 Main Results

The uniqueness result will be a consequence of the following generalized maximum principle and the next lemmas.

THEOREM 1 ([4]). Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy the inequality $Lu + \gamma(x)u \geq 0$ in $\Omega$, where $\gamma \geq 0$ in $\Omega$. Suppose that

$$\sup_{\Omega} \gamma < \frac{4n + 4}{(\text{diam } \Omega)^2}$$

(3)

holds. Then, the function $u/w_1$ satisfies a generalized maximum principle in $\Omega$, i.e., either the function $u/w_1$ assumes its maximum value on $\partial \Omega$ or is constant in $\Omega$. Here

$$w_1(x) = 1 - \alpha(x_1^2 + \cdots + x_n^2) \in C^\infty(\mathbb{R}^n)$$

and $\alpha = \sup_{\Omega} \gamma/2n$.

If $\Omega$ lies in strip of width $d$ and if we impose the restriction

$$\sup_{\Omega} \gamma < \frac{\pi^2}{d^2},$$

(4)

we obtain that $u/w_2$ satisfies a generalized maximum principle in $\Omega$. Here

$$w_2 = \cos \left( \frac{\pi}{2(d + \varepsilon)} \right) \prod_{j=1}^n \cosh(\varepsilon x_j) \in C^\infty(\overline{\Omega}),$$

for some $i \in \{1, \ldots, n\}$, where $\varepsilon > 0$ is small.

For simplicity, we shall consider only the case when $m$ is even, i.e., we shall deal with the equation

$$\Delta^m u - a_{m-1}(x)\Delta^{m-1} u + \cdots + a_0(x)u = 0 \quad \text{in } \Omega.$$  

(5)

Similar results will hold if $m$ is odd.

LEMMA 1. Let $u$ be a classical solution of (5). Let

$$P_1 = \frac{1}{2} (\Delta^{m-1} u)^2 + \frac{a_{m-1}}{2} (\Delta^{m-2} u)^2 + (\Delta^{m-3} u)^2 + \cdots + u^2.$$ 

Suppose that $a_{m-3}, \ldots, a_1 \geq 0$, $a_2, a_0 > 0$ and $\Delta(1/a_{m-2}) \leq 0$ in $\Omega$. If one of the following conditions is satisfied

(a) $$4a_{m-1} - a_{m-3} - a_{m-4} - \cdots - a_0 \geq 0 \quad \text{in } \Omega$$

(6)

and

$$A = \max \left\{ 1 + \sup_{\Omega} a_0, 2 + \sup_{\Omega} a_1, \ldots, 2 + \max_{\Omega} \frac{a_{m-2}}{2} \right\} < \frac{4n + 4}{(\text{diam } \Omega)^2},$$

(7)

(b) $$a_{m-1} \geq 0 \quad \text{in } \Omega$$

(8)
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and

\[
\max \left\{ A, \sup_{\Omega} \frac{a_{m-3} + \cdots + a_{0}}{2} \right\} < \frac{4n + 4}{(\text{diam } \Omega)^2},
\]

then either the function \( P_{1} / w_{1} \) assumes its maximum value on \( \partial \Omega \) or is constant in \( \overline{\Omega} \).

**PROOF.** A computation (using equation (5)) shows that in \( \Omega \),

\[
\frac{1}{2} \Delta \left( (\Delta^{m-1}u)^2 \right) \geq \Delta^{m-1}u \Delta^m u = a_{m-1}(\Delta^{m-1}u)^2 - a_{m-2}\Delta^{m-2}u \Delta^{m-1}u - a_{m-3}\Delta^{m-3}u \Delta^{m-1}u - \cdots - a_{0}u \Delta^{m-1}u.
\]

From the inequalities

\[
(-1)^{i}a_{i-3}\Delta^{i-3}u \Delta^{m-1}u \geq -\frac{a_{i-3}}{4}(\Delta^{m-1}u)^2 - a_{i-3}(\Delta^{i-3}u)^2, \quad i = 3, \ldots, m,
\]

and

\[
\frac{1}{2} \Delta \left( a_{m-2}(\Delta^{m-2}u)^2 \right) \geq a_{m-2}\Delta^{m-1}u \Delta^{m-2}u,
\]

we get

\[
\frac{1}{2} \Delta \left( ((\Delta^{m-1}u)^2 + a_{m-2}(\Delta^{m-2}u)^2) \right) \geq \frac{1}{2} \Delta \left( (\Delta^{m-1}u)^2 + a_{m-2}(\Delta^{m-2}u)^2) \right) \geq (a_{m-1} - a_{m-3}/4 - a_{m-4}/4 - \cdots - a_{0}/4)(\Delta^{m-1}u)^2
\]

\[
- a_{m-3}(\Delta^{m-3}u)^2 - a_{m-4}(\Delta^{m-4}u)^2 - \cdots - a_{0}u^2.
\]

Since

\[
\Delta \left( (\Delta^{m-3}u)^2 \right) \geq 2\Delta^{m-2}u \Delta^{m-3}u \geq -(\Delta^{m-2}u)^2 - (\Delta^{m-3}u)^2,
\]

\[
\Delta \left( (\Delta^{m-4}u)^2 \right) \geq 2\Delta^{m-3}u \Delta^{m-4}u \geq -(\Delta^{m-3}u)^2 - (\Delta^{m-4}u)^2,
\]

\[
\cdots,
\]

\[
\Delta u^2 \geq 2u \Delta u \geq -\Delta u^2 - u^2,
\]

we deduce that \( P_{1} \) satisfies the differential inequality

\[
\Delta P_{1} \geq (a_{m-1} - a_{m-3}/4 - a_{m-4}/4 - \cdots - a_{0}/4)(\Delta^{m-1}u)^2 - (\Delta^{m-2}u)^2
\]

\[
- (2 + a_{m-3})(\Delta^{m-3}u)^2 - \cdots - (2 + a_{1})(\Delta u)^2 - (1 + a_{0})u^2.
\]

Hence

\[
\Delta P_{1} + \gamma P_{1} \geq 0 \quad \text{in } \Omega,
\]

where

\[
\gamma = \max \left\{ 1 + \sup_{\Omega} a_{0}, 2 + \sup_{\Omega} a_{1}, \ldots, 2 + \sup_{\Omega} a_{m-3}, \max_{\Omega} \left\{ 1, \sup_{\Omega} a_{m-2}/2 \right\} \right\}.
\]

By (7) we have

\[
\gamma < \frac{4n + 4}{(\text{diam } \Omega)^2}.
\]
Now the proof of (a) follows from Theorem 1. The proof for (b) is similar.

**Lemma 2.** Let \( u \) be a classical solution of (5). Let

\[
P_2 = \frac{1}{2}(\Delta^{m-1}u)^2 + (\Delta^{m-2}u)^2 + (\Delta^{m-3}u)^2 + \cdots + u^2.
\]

Suppose that \( a_{m-1}, \ldots, a_1 \geq 0 \) and \( a_0 > 0 \) in \( \Omega \). If

\[
\max \left\{ \sup_{\Omega} \frac{a_0}{2} + \sup_{\Omega} \frac{a_1}{2}, \ldots, 2 + \sup_{\Omega} \frac{a_{m-2}}{2}, A_1 \right\} < \frac{4n + 4}{(\text{diam } \Omega)^2}, \tag{11}
\]

where \( A_1 = \max\{1 + \sup_{\Omega} a_0, 2, \sup_{\Omega} a_1, \ldots, 2 + \sup_{\Omega} a_{m-2}\} \), then either the function \( P_2/w_1 \) assumes its maximum on \( \partial \Omega \) or is a constant in \( \Omega \).

**Proof.** As in the proof of Lemma 1, we get

\[
\frac{1}{2} \Delta (\Delta^{m-1}u)^2 \geq \Delta^{m-1}u \Delta^m u
\]

\[
= a_{m-1}(\Delta^{m-1}u)^2 - a_{m-2}\Delta^{m-2}u \Delta^{m-1}u - \cdots - a_0u \Delta^{m-1}u.
\]

Since

\[
-a_0u \Delta^{m-1}u \geq -\frac{a_0}{4}(\Delta^{m-1}u)^2 - a_0u^2,
\]

\[
-\cdots - a_{m-2}u \Delta^{m-1}u \Delta^{m-2}u \geq -\frac{a_{m-2}}{4}(\Delta^{m-1}u)^2 - a_{m-2}(\Delta^{m-2}u)^2,
\]

and

\[
\Delta (\Delta^{m-2}u)^2 \geq -(\Delta^{m-2}u)^2 - (\Delta^{m-1}u)^2,
\]

\[
\cdots,
\]

\[
\Delta u^2 \geq -\Delta u^2 - u^2,
\]

we get that

\[
\Delta P_2 \geq -(1 + a_{m-2}/4 + a_{m-3}/4 + \cdots + a_{1}/4 + a_0/4)(\Delta^{m-1}u)^2 - (2 + a_{m-2})(\Delta^{m-2}u)^2
\]

\[
- (2 + a_{m-3})(\Delta^{m-3}u)^2 - \cdots - (2 + a_1)(\Delta u)^2 - (1 + a_0)u^2.
\]

Hence

\[
\Delta P_2 + \gamma P_2 \geq 0 \quad \text{in } \Omega,
\]

where

\[
\gamma = \max \{A_1, \{\sup_{\Omega} a_0/2 + \sup_{\Omega} a_1/2 + \cdots + \sup_{\Omega} a_{m-2}/2 + 2\}\}.
\]

**Lemma 3.** Let \( u \) be a classical solution of (5). Suppose that \( a_{m-2}, \ldots, a_0 \geq 0 \) in \( \Omega \). If one of the following conditions is fulfilled

(a)

\[
\max \{1 + \sup_{\Omega} a_0^2, 2 + \sup_{\Omega} a_1^2, \ldots, 2 + \sup_{\Omega} a_{m-2}^2\} < \frac{4n + 4}{(\text{diam } \Omega)^2} \tag{12}
\]
and \(4a_{m-1} \geq m + 3\) in \(\Omega\); or

(b) 

\[
\max \left\{ 1 + \sup_{\Omega} a_0^2, 2 + \sup_{\Omega} a_1^2, \ldots, 2 + \sup_{\Omega} a_{m-2}^2, 2 + \frac{m-1}{2} \right\} < \frac{4n + 4}{(\text{diam } \Omega)^2} \tag{13}
\]

and \(a_{m-1} \geq 0\) in \(\Omega\),

then either the function \(P_2/w_1\) assumes its maximum on \(\partial \Omega\) or is a constant in \(\overline{\Omega}\).

This may be proved exactly as Lemma 2, except the inequalities (10) are replaced by

\[
(-1)^i a_{i-3} \Delta^{i-3} u \Delta^{m-1} u \geq -\frac{1}{4}(\Delta^{m-1} u)^2 - a_{i-3}^2 (\Delta^{i-3} u)^2, \quad i = 3, \ldots, m.
\]

It is clear that Lemma 3 remains valid if the coefficients \(a_{m-2}, \ldots, a_0\) have arbitrary sign in \(\Omega\).

The following particular result becomes sharper than Lemma 2 if we choose \(a_0\) and \(a_1\) appropriately.

**Lemma 4.** Let \(u\) be a classical solution of (5). Let

\[
P_3 = \frac{1}{2}(\Delta^{m-1} u - a_1 u)^2 + P_2.
\]

Suppose that \(a_{m-1} = \cdots = a_2 = 0\) and \(a_0 > 0\) in \(\Omega\). If \(a_1 \equiv \text{constant} > 0\) and if

\[
\max \left\{ 2 + 2 \sup_{\Omega} \frac{a_0}{a_1} + 2a_1, 2 + \frac{a_1}{4} \right\} < \frac{4n + 4}{(\text{diam } \Omega)^2}, \tag{14}
\]

then, the function \(P_3/w_1\) assumes its maximum on \(\partial \Omega\) or is a constant in \(\overline{\Omega}\).

**Proof.** A calculation gives

\[
\Delta \left( \frac{1}{2}(\Delta^{m-1} u - a_1 u)^2 + \frac{1}{2}(\Delta^{m-1} u)^2 \right) \\
\geq -2a_0 u \Delta^{m-1} u + a_1 \Delta u \Delta^{m-1} u + a_0 a_1 u^2 \\
= a_0 a_1 \left( u^2 - \frac{2}{a_1} u \Delta^{m-1} u + \frac{1}{a_1^2} (\Delta^{m-1} u)^2 \right) - \frac{a_0}{a_1} (\Delta^{m-1} u)^2 + a_1 \Delta u \Delta^{m-1} u \\
\geq -\frac{a_0}{a_1} (\Delta^{m-1} u)^2 - \frac{a_1}{4}(\Delta u)^2 - a_1 (\Delta^{m-1} u)^2
\]

in \(\Omega\). It follows that

\[
\Delta P_3 \geq - \left( \frac{a_0}{a_1} + a_1 + 1 \right) (\Delta^{m-1} u)^2 - 2(\Delta^{m-2} u)^2 - \cdots - 2(\Delta^3 u)^2 - \\
- \left( \frac{a_1}{4} + 2 \right) (\Delta u)^2 - u^2
\]

in \(\Omega\). Hence

\[
\Delta P_3 + \gamma P_3 \geq 0 \quad \text{in } \Omega,
\]
where $\gamma = \max\{2 + 2\sup_{\Omega}(a_0/a_1) + 2a_1, 2 + a_1/4\}$.

We now state our main result.

**THEOREM 2.** There is at most one classical solution of the boundary value problem (2) provided the coefficients $a_{m-1}, \ldots, a_0$ satisfy the conditions imposed in Lemma 1, Lemma 2, Lemma 3 or Lemma 4.

**PROOF.** Suppose that the hypothesis of Lemma 1 is satisfied. Define $u = u_1 - u_2$, where $u_1$ and $u_2$ are solutions of (2). Then $u_1$ and $u_2$ satisfy the equation (5) and

$$u = \Delta u = \cdots = \Delta^{m-1} u = 0 \quad \text{on } \partial \Omega. \quad (15)$$

Hence, by Theorem 1 either

i). there exists a constant $k \in \mathbb{R}$ such that

$$\frac{P_1}{w_1} = k \quad \text{in } \Omega, \quad (16)$$

or

ii). $P_1/w_1$ does not attain a maximum in $\Omega$.

Case i). By continuity (16) holds in $\overline{\Omega}$. By the boundary conditions (15) we obtain $P_1 = 0$ on $\partial \Omega$, i.e., $k = 0$. It follows that $P_1 \equiv 0$ in $\Omega$, which means $u \equiv 0$ in $\Omega$. Hence $u_1 = u_2$ in $\Omega$.

Case ii). From

$$\max_{\Omega} \frac{P_1}{w_1} = \max_{\partial \Omega} \frac{P_1}{w_1}$$

and (15) we get

$$0 \leq \max_{\Omega} \frac{P_1}{w_1} = 0,$$

i.e., $u_1 = u_2$ in $\Omega$.

We can argue similarly if we are under the hypotheses of Lemma 2, Lemma 3 or Lemma 4. The proof is complete.

Of course, our method can also be applied to the problem (1) to get results in arbitrary domains $\Omega$.

Next, we consider classical solutions of the equation

$$\Delta^4 u + a_2(x)\Delta^2 u - a_3(x)\Delta u + a_4(x)u = 0 \quad \text{in } \Omega. \quad (17)$$

**LEMMA 5.** Let $u$ be a classical solution of (17). Assume that

$$a_2 > 0, \quad \Delta(1/a_2) \leq 0 \quad \text{in } \Omega, \quad (18)$$

$$a_4 > 0, \quad \Delta(1/a_4) \leq 0 \quad \text{in } \Omega, \quad (19)$$

$$a_2 - 2a_4 - 1 > 0, \quad \Delta(1/(a_2 - 2a_4 - 1)) \leq 0 \quad \text{in } \Omega. \quad (20)$$

If

$$\max \left\{ \sup_{\Omega} a_3, \sup_{\Omega} \frac{1}{a_4}, \sup_{\Omega} \frac{a_4^2}{a_2 - 2a_4 - 1} \right\} < \frac{2n + 2}{(\text{diam } \Omega)^2}, \quad (21)$$
then, the function $P_4/w_1$ assumes its maximum on $\partial \Omega$ or is a constant in $\overline{\Omega}$. Here

$$P_4 = \frac{1}{2}(\Delta^3 u + \Delta u)^2 + a_4(\Delta^2 u + u)^2 + \frac{a_2 - 2a_4 - 1}{2} (\Delta^2 u)^2 + \frac{a_2 + 1}{2} (\Delta u)^2$$

$$+ \frac{1}{2} (\Delta^3 u)^2 + \frac{a_2}{2} (\Delta^2 u)^2 + \frac{1}{2} a_4 u^2.$$ 

Under the hypotheses of Lemma 5, an uniqueness result follows for problem (1). We note that this uniqueness result is not a particular result of Theorem 2. Moreover we do not impose any convexity assumption on $\partial \Omega$.

Finally, we give an application of the uniqueness result that follows from Lemma 5. We see that the boundary value problem

$$\begin{cases} 
\Delta^4 u + 4(x^2 + y^2 + 3)\Delta^2 u - ((x^2 + y^2 + 3)^2/4)\Delta u + (x^2 + y^2 + 3) u = 0 & \text{in } \Omega \\
u = 13/4, \Delta u = 4, \Delta^2 u = 0, \Delta^3 u = 0 & \text{on } \partial \Omega, 
\end{cases}$$

has the solution $u(x, y) = x^2 + y^2 + 3$ in $\Omega = \{(x, y) | x^2 + y^2 \leq 1/4\}$.

Since (18), (19), (20) and (21) are satisfied, we get by the uniqueness result that follows from Lemma 5 that $u(x, y) = x^2 + y^2 + 3$ is the unique solution.

As our final remarks, for some domains we may improve the maximum principle, i.e. the constant $C(n, \text{diam } \Omega) = (4n + 4)/(\text{diam } \Omega)$ can be taken larger (see for details [2] and [3]).

References


