Convexity Properties And Inequalities For A Generalized Gamma Function

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Abstract

For the $\Gamma_p$-function, defined by Euler, are given some properties related to convexity and log-convexity. Also, some properties of $p$ analogue of the $\psi$ function have been established. The $p$-analogue of some inequalities from [6] and [7] have been proved. As an application, when $p \to \infty$, we obtain all results of [6].

1 Introduction

In this section we will present definitions used in this paper. The Euler gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$ 

The digamma (or psi) function is defined for positive real numbers $x$ as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{x} \sum_{n \geq 1} \frac{x}{n(n+x)}, \quad (1)$$

where $\gamma = 0.57721...$ denotes Euler's constant.

Euler, gave another equivalent definition for the $\Gamma(x)$ (see [2],[5])

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1) \cdots (x+p)} = \frac{p^x}{x(1 + \frac{x}{p}) \cdots (1 + \frac{x}{p})}, \quad x > 0 \quad \text{and} \quad p \geq 1 \quad \text{is a positive integer,} \quad \text{and} \quad \Gamma(x) = \lim_{p \to \infty} \Gamma_p(x). \quad \text{(3)}$$

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We define the $p$-analogue of the psi function as the logarithmic derivative of the $\Gamma_p$ function, that is
\[ \psi_p(x) = \frac{d}{dx} \ln \Gamma_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)}. \] (4)

**DEFINITION 1.1.** The function $f$ is called log-convex if for all $\alpha, \beta > 0$ such that $\alpha + \beta = 1$ and for all $x, y > 0$ the following inequality holds
\[ \log f(\alpha x + \beta y) \leq \alpha \log f(x) + \beta \log f(y) \]
or equivalently
\[ f(\alpha x + \beta y) \leq (f(x))^\alpha \cdot (f(y))^\beta. \]

**DEFINITION 1.2.** Let $f : I \subseteq (0, \infty) \rightarrow (0, \infty)$ be a continuous function. Then $f$ is called geometrically convex on $I$ if there exists $n \geq 2$ such that one of the following two inequalities holds:
\[ f(\sqrt{x_1x_2}) \leq \sqrt{f(x_1)f(x_2)} \] (5)
\[ f\left( \prod_{i=1}^{n} x_i^{\lambda_i} \right) \leq \prod_{i=1}^{n} (f(x_i))^{\lambda_i} \] (6)
where $x_1, \ldots, x_n \in I; \lambda_1, \ldots, \lambda_n > 0$ with $\sum_{i=1}^{n} \lambda_i = 1$. If inequalities (5) and (6) are reversed, then $f$ is called geometrically concave function on $I$.

In the next section, we derive several convexity and log-convexity properties related the $\Gamma_p$.

### 2 Some Properties of $\Gamma_p$

We begin with recurrent relations for $\Gamma_p$ and $\psi_p$.

**LEMMA 2.1.** Let $\Gamma_p$ be defined as in (2). Then
\[ \Gamma_p(x+n) = p^n \cdot \prod_{i=0}^{n-1} \frac{(x+i)}{(x+p+i)} \Gamma_p(x), \quad x+n > 0. \] (7)

**PROOF.** Using (2) one finds that:
\[ \frac{\Gamma_p(x+n)}{\Gamma_p(x+n-1)} = \frac{x+n-1}{p^{-1}(x+n+p)} \]
Hence
\[ \Gamma_p(x+n) = \frac{p(x+n-1)}{(x+n+p)} \cdot \Gamma_p(x+n-1). \]

In a similar way, we have:
\[ \Gamma_p(x+n-1) = \frac{p(x+n-2)}{(x+(n-1)+p)} \cdot \Gamma_p(x+n-2) \]
It means
\[ \Gamma_p(x + n) = \frac{p^2(x + n - 1)(x + n - 2)}{(x + n + p)(x + (n - 1) + p)} \cdot \Gamma_p(x + n - 2). \]

Continuing in this way we obtain:
\[ \Gamma_p(x + n) = \frac{p^n(x + n - 1)(x + n - 2) \ldots x}{(x + n + p)(x + p + n - 1) \ldots (x + p + 1)} \cdot \Gamma_p(x), \]
completing the proof.

**Remark 2.2.** When \( p \to \infty \), we obtain the well known relation
\[ \Gamma(x) = \frac{\Gamma(x + n)}{x(x + 1) \ldots (x + n - 1)}, \quad x + n > 0. \]

**Lemma 2.3.** a) The function \( \psi_p \) defined by (4) has the following series representation
\[ \psi_p(x) = \ln p - \sum_{k=0}^{p} \frac{1}{x + k}. \]

b) The function \( \psi_p \) is increasing on \((0, \infty)\).

c) The function \( \psi_p' \) is strictly completely monotonic on \((0, \infty)\).

**Proof.** a) By (2) we have:
\[
\psi_p(x) = \frac{d}{dx} \left( \ln \Gamma_p(x) \right) \\
= \frac{d}{dx} \left( x \ln p - \left( \ln x + \ln(1 + x) + \ln \left( 1 + \frac{x}{2} + \ldots + \ln \left( 1 + \frac{x}{p} \right) \right) \right) \right) \\
= \ln p - \left( \frac{1}{x + 1 + x} + \frac{1}{1 + \frac{x}{2}} + \ldots + \frac{1}{1 + \frac{x}{p}} \right) \\
= \ln p - \sum_{k=0}^{p} \frac{1}{x + k}. 
\]

b) Let \( 0 < x < y \). Using (8) we obtain
\[
\psi_p(x) - \psi_p(y) = - \sum_{k=0}^{p} \frac{1}{x + k} + \sum_{k=0}^{p} \frac{1}{y + k} = \sum_{k=0}^{p} \frac{(x - y)}{(x + k)(y + k)} < 0. 
\]

c) Deriving \( n \) times the relation (8) one finds that:
\[ \psi_p^{(n)}(x) = \sum_{k=0}^{p} \frac{(-1)^{n-1} \cdot n!}{(x + k)^{n+1}}, \]
hence \( (-1)^n(\psi_p'(x))^{(n)} > 0 \) for \( x > 0, n \geq 0 \).

**Remark 2.4.** We note that \( \lim_{p \to \infty} \psi_p^{(n)}(x) = \psi^{(n)}(x) \).
By (8) one has the following:

**COROLLARY 2.5.**

\[
\psi_p(x + 1) = \frac{1}{x} - \frac{1}{x + p + 1} + \psi_p(x).
\]

**COROLLARY 2.6.** The function \(\log \Gamma_p(x)\) is convex for \(x > 0\).

**PROOF.** Taking \(n = 2\) in (9) we have

\[
\psi'_p(x) = \sum_{k=0}^{p} \frac{1}{(x + k)^2}.
\]

(10)

So, for \(x > 0\), \(\psi'_p(x) > 0\) hence \(\psi_p\) is a monotonous function on the positive axis and therefore the function \(\log \Gamma_p(x)\) is convex for \(x > 0\).

**LEMMA 2.7.** Let \(\psi_p\) be as in (8). Then

\[
\lim_{p \to \infty} \psi_p(x) = \psi(x).
\]

(11)

**PROOF.** By (8) we have:

\[
\lim_{p \to \infty} \psi_p(x) = \lim_{p \to \infty} \ln p - \lim_{p \to \infty} \frac{1}{x} \left( \frac{1}{x} + \sum_{k=1}^{p} \frac{1}{x + k} \right)
\]

\[
= \lim_{p \to \infty} \left( \ln p - 1 - \frac{1}{2} - \ldots - \frac{1}{p} \right) - \lim_{p \to \infty} \frac{1}{x} \left( \sum_{k=1}^{p} \frac{1}{x + k} - \sum_{k=1}^{p} \frac{1}{k} \right)
\]

\[
= -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k(k + x)}
\]

\[
= \psi(x).
\]

**THEOREM 2.8.** The function

\[
\Gamma_p(x) = \frac{p^x}{x(1 + \frac{x}{1}) \ldots (1 + \frac{x}{p})}, x > 0
\]

is log-convex.

**PROOF.** We have to prove that for all \(\alpha, \beta > 0, \alpha + \beta = 1, x, y > 0\)

\[
\log \Gamma_p(\alpha x + \beta y) \leq \alpha \log \Gamma_p(x) + \beta \log \Gamma_p(y)
\]

(12)

which is equivalent to

\[
\Gamma_p(\alpha x + \beta y) \leq (\Gamma_p(x))^{\alpha} \cdot (\Gamma_p(y))^{\beta}.
\]

(13)

By Young’s inequality (see [3]) we have:

\[
x^\alpha \cdot y^\beta \leq \alpha x + \beta y.
\]

(14)
From (14) we obtain:

\[
(1 + \frac{x}{k})^\alpha \cdot (1 + \frac{y}{k})^\beta \leq \alpha \left(1 + \frac{x}{k}\right) + \beta \left(1 + \frac{y}{k}\right) = 1 + \frac{\alpha x + \beta y}{k}
\]

(15)

for all \(k \geq 1, k \in \mathbb{N}\).

Multiplying (15) for \(k = 1, 2, \ldots, p\) one obtains

\[
\left(1 + \frac{x}{1}\right)^\alpha \cdots \left(1 + \frac{x}{p}\right)^\alpha \cdot \left(1 + \frac{y}{1}\right)^\beta \cdots \left(1 + \frac{y}{p}\right)^\beta \leq \left(1 + \frac{\alpha x + \beta y}{1}\right) \cdots \left(1 + \frac{\alpha x + \beta y}{p}\right).
\]

Now, taking the reciprocal values and multiplying by \(p^{\alpha x + \beta y}\) one obtains (13) and thus the proof is completed.

For the proof of the following result see [5].

**Proposition 2.9.** Let \(f\) be a log-convex function on \((0, \infty)\). Then the function \(F_a\) given by

\[
F_a(x) = a^x f(x)
\]

is convex for any \(a > 0\).

From Proposition 2.9 and Theorem 2.8 immediately follows the following corollary.

**Corollary 2.10.** The functions \(F_a, G_a\) given by

\[
F_a(x) = a^x \Gamma_p(x), x > 0; G_a(x) = x^a \Gamma_p(x), x > 0,
\]

respectively, are convex.

Another easily established property related to \(\psi_p\) is the following proposition.

**Proposition 2.11.** The function \(x \mapsto x\psi_p(x), x > 0\) is strictly convex.

**Proof.** We have

\[
(x\psi_p(x))' = \psi_p(x) + x\psi'_p(x)
\]

\[
(x\psi_p(x))'' = 2\psi'_p(x) + x\psi''_p(x).
\]

Using (9) we obtain

\[
(x\psi_p(x))'' = 2 \sum_{k=0}^{p} \frac{1}{(x+k)^2} - 2 \sum_{k=0}^{p} \frac{x}{(x+k)^3} = 2 \sum_{k=0}^{p} \frac{k}{(x+k)^3} > 0.
\]

Next we will prove a result on geometric convexity related to \(\Gamma_p\) that will be used in the next section.

For the proof of the following Lemma see [4].

**Lemma 2.12.** Let \((a, b) \subset (0, \infty)\) and \(f : (a, b) \rightarrow (0, \infty)\) be a differentiable function. Then \(f\) is geometrically convex if and only if the function \(\frac{f(x)}{f'(x)}\) is nondecreasing.

**Theorem 2.13.** The function \(f(x) = e^x \cdot \Gamma_p(x)\) is geometrically convex.

**Proof.** Let \(f(x) = e^x \cdot \Gamma_p(x)\). Then \(\ln f(x) = x + \ln \Gamma_p(x)\). Hence

\[
\frac{f'(x)}{f(x)} = 1 + \psi_p(x).
\]
So, \( x \frac{f'(x)}{f(x)} = x + x \psi_p(x) \). Let \( \theta(x) = x + x \psi_p(x) \). Then we have

\[
\theta'(x) = 1 + \psi_p(x) + x \psi'_p(x).
\]

Using (8) and (9) one obtains

\[
\theta'(x) = 1 + \ln p - \psi_p(x) - \psi_p'(x) = 1 + \ln p - \sum_{k=1}^{p} \frac{k}{(x+k)^2}.
\]

Let \( v(x) = 1 + \ln p - \sum_{k=1}^{p} \frac{k}{(x+k)^2} \). One can easily show that for \( x > 0 \) the function \( v \) is nondecreasing. Hence, \( v(x) > v(0) \). On the other side

\[
v(0) = 1 - \left( \sum_{k=1}^{p} \frac{1}{k} - \ln p \right) \geq 0.
\]

Hence \( \theta'(x) > 0 \) so \( \theta \) is nondecreasing.

**REMARK 2.14.** Using similar approach, one can show that the function \( f(x) = \frac{e^{x \Gamma_p(x)}}{x^a}, a \neq 0 \), is geometrically convex.

**REMARK 2.15.** In [7], it is proved that the function \( f(x) = \frac{e^{x \Gamma_p(x)}}{x^a} \) is geometrically convex.

In relation to the function \( f_1(x) = \frac{e^{x \Gamma_p(x)}}{x^a} \), one can show that it is geometrically convex in the neighborhood of zero, and it is not geometrically convex for \( x > p \), while for the rest the proof could not be established.

### 3 Inequalities and Applications

In this section we prove some inequalities related to \( \Gamma_p \) function. Some applications of \( \Gamma_p \) are presented at the end of the section.

**LEMMA 3.1.** Let \( x > 1 \). Then

\[
\gamma + \ln p + \psi(x) - \psi_p(x) > 0.
\]

**PROOF.** Using the series representations of the functions \( \psi \) and \( \psi_p \) we obtain:

\[
\gamma + \ln p + \psi(x) - \psi_p(x) = (x - 1) \sum_{k=0}^{\infty} \frac{1}{(1+k)(x+k)} + \sum_{k=0}^{p} \frac{1}{(x+k)} > 0.
\]

Using previous Lemma we have:
LEMMA 3.2. Let $a$ be a positive real number such that $a + x > 1$. Then
$$\gamma + \ln p + \psi(x + a) - \psi_p(x + a) > 0.$$  

THEOREM 3.3. Let $f$ be a function defined by
$$f(x) = \frac{e^{\gamma x} \Gamma(x + a)}{p^{-x} \Gamma_p(x + a)}, \quad x \in (0, 1)$$  
where $a, b$ are real numbers such that $a + x > 1$. If $\psi(x + a) > 0$ or $\psi_p(x + a) > 0$ then the function $f$ is increasing for $x \in (0, 1)$ and the following double inequality holds
$$\frac{\Gamma(a)}{p^x \cdot e^{\gamma x} \Gamma_p(a)} < \frac{\Gamma(x + a)}{\Gamma_p(x + a)} < p^{1-x} \cdot e^{\gamma(1-x)} \cdot \frac{\Gamma(1 + a)}{\Gamma_p(1 + a)}. \quad (18)$$  

PROOF. Let $g$ be a function defined by $g(x) = \ln f(x)$ for $x \in (0, 1)$. Then
$$g(x) = \gamma x + \ln \Gamma(x + a) + x \ln p - \ln \Gamma_p(x + a).$$  
Then
$$g'(x) = \gamma + \ln p + \psi(x + a) - \psi_p(x + a).$$  
By Lemma 18 we have $g'(x) > 0$. It means that $g$ is increasing on $(0, 1)$. This implies that $f$ is increasing on $(0, 1)$ so we have $f(0) < f(x) < f(1)$ and the result follows.

For the proof of the following Lemma see [4].

LEMMA 3.4. Let $(a, b) \subset (0, \infty)$ and $f : (a, b) \rightarrow (0, \infty)$ be a differentiable function. Then $f$ is geometrically convex if and only if the inequality
$$\frac{f(x)}{f(y)} \geq \left( \frac{x}{y} \right)^{\frac{f'(x)}{f'(y)}} \quad (19)$$  
holds for any $x, y \in (a, b)$.

The following result is the analogue of the Theorem 1.2 from [7].

THEOREM 3.5. For $x > 0, y > 0$ the double inequality holds
$$\left( \frac{x}{y} \right)^{y(1+\psi_p(y))} \cdot e^{y-x} \leq \frac{\Gamma_p(x)}{\Gamma_p(y)} \leq \left( \frac{x}{y} \right)^{x(1+\psi_p(x))} \cdot e^{y-x}. \quad (20)$$  

PROOF. Combination of Theorem 2.13, Lemma 3.4 and relation (16) leads to:
$$\frac{e^{x} \Gamma_p(x)}{e^{y} \Gamma_p(y)} \geq \left( \frac{x}{y} \right)^{y(1+\psi_p(y))}$$  
and
$$\frac{e^{y} \Gamma_p(y)}{e^{x} \Gamma_p(x)} \geq \left( \frac{y}{x} \right)^{x(1+\psi_p(x))}.$$  
Hence the inequality (20) is established.
In the following, we give the $\Gamma_p$ analogue of results from [6]. Since the proofs are almost similar, we omit them.

**LEMMA 3.6.** Let $a, b, c, d, e$ be real numbers such that $a + bx > 0$, $d + ex > 0$ and $a + bx \leq d + ex$. Then

$$\psi_p(a + bx) - \psi_p(d + ex) \leq 0.$$  (21)

**LEMMA 3.7.** Let $a, b, c, d, e, f$ be real numbers such that $a + bx > 0$, $d + ex > 0$, $a + bx \leq d + ex$ and $ef \geq bc > 0$. If (i) $\psi_p(a + bx) > 0$, or (ii) $\psi_p(d + ex) > 0$, then

$$bc\psi_p(a + bx) - ef\psi_p(d + ex) \leq 0.$$  (22)

**LEMMA 3.8.** Let $a, b, c, d, e, f$ be real numbers such that $a + bx > 0$, $d + ex > 0$, $a + bx \leq d + ex$ and $bc \geq ef > 0$. If (i) $\psi_p(d + ex) < 0$, or (ii) $\psi_p(a + bx) < 0$, then

$$bc\psi_p(a + bx) - ef\psi_p(d + ex) \leq 0.$$  (23)

**THEOREM 3.9.** Let $f_1$ be a function defined by

$$f_1(x) = \frac{\Gamma_p(a + bx)^c}{\Gamma_p(d + ex)^f}, \quad x \geq 0$$  (24)

where $a, b, c, d, e, f$ are real numbers such that: $a + bx > 0$, $d + ex > 0$, $a + bx \leq d + ex$, $ef \geq bc > 0$. If $\psi_p(a + bx) > 0$ or $\psi_p(d + ex) > 0$ then the function $f_1$ is decreasing for $x \geq 0$ and for $x \in [0, 1]$ the following double inequality holds:

$$\frac{\Gamma_p(a + b)^c}{\Gamma_p(d + e)^f} \leq \frac{\Gamma_p(a + bx)^c}{\Gamma_p(d + ex)^f} \leq \frac{\Gamma_p(a)^c}{\Gamma_p(d)^f}.$$  (25)

In a similar way, using Lemma 3.8, it is easy to prove the following Theorem.

**THEOREM 3.10.** Let $f_1$ be a function defined by

$$f_1(x) = \frac{\Gamma_p(a + bx)^c}{\Gamma_p(d + ex)^f}, \quad x \geq 0,$$  (26)

where $a, b, c, d, e, f$ are real numbers such that: $a + bx > 0$, $d + ex > 0$, $a + bx \leq d + ex$, $bc \geq ef > 0$. If $\psi_p(d + ex) < 0$ or $\psi_p(a + bx) < 0$ then the function $f_1$ is decreasing for $x \geq 0$ and for $x \in [0, 1]$ the inequality (25) holds.

At the end we provide some applications related to the $\Gamma_p$ function.

**REMARK 3.11.** Using (2) and (3) and the fact that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ we obtain the following representation for $\pi$

$$\sqrt{\pi} = \lim_{p \to \infty} \frac{\sqrt{p}}{\frac{1}{2}\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right)\cdots\left(1 + \frac{1}{2p}\right)}.$$  (27)

**REMARK 3.12.** Using (3) in equations (25) and (26) we obtain all the results of [6].
References


