Perron Complements Of Diagonally Dominant Matrices And H-Matrices*

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Abstract

In this paper, we consider properties of the Perron complements of diagonally dominant matrices and H-matrices.

1 Introduction

Let $A = (a_{ij})$ be an $n \times n$ matrix, and recall that $A$ is (row) diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \quad i = 1, 2, \ldots, n. \quad (1)$$

$A$ is further said to be strictly diagonally dominant if all the strict inequalities in (1) hold. Obviously the principal submatrices of strictly diagonally dominant matrices are strictly diagonally dominant and thus $A$ is nonsingular.

For $A \in \mathbb{Z} = \{ (a_{ij}) \in \mathbb{R}^{n,n} : a_{ij} \leq 0, i \neq j \}$, if $A = aI - B, B \geq 0, a > \rho(B)$, then $A$ is called an M-matrix. The comparison matrix $\mu(A) = (\mu_{ij})$ is defined by

$$\mu_{ij} = \begin{cases} -|a_{ij}| & i \neq j, \\ |a_{ij}| & i = j. \end{cases}$$

$A \in \mathbb{C}^{n,n}$ is called an H-matrix if $\mu(A)$ is an M-matrix. If there exists a positive diagonal matrix $D = \text{diag} (d_1, \ldots, d_n)$ such that $D^{-1}AD$ is strictly diagonally dominant, we call $A$ a generalized diagonally dominant matrix. It is well-known that $A$ is an H-matrix is equivalent to $A$ is generalized diagonally dominant.

The empty set is denoted by $\phi$. Let $\alpha, \beta$ be nonempty ordered subsets of $\langle n \rangle := \{1, 2, \ldots, n\}$, both consisting of strictly increasing integers. By $A(\alpha, \beta)$ we shall denote the submatrix of $A$ lying in rows indexed by $\alpha$ and columns indexed by $\beta$. If, in addition, $\alpha = \beta$, then the principal submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$.

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Suppose that $\alpha \subset \langle n \rangle$. If $A(\alpha)$ is nonsingular, then the Schur complement of $A(\alpha)$ in $A$ is given by

$$S(A/A(\alpha)) = A(\beta) - A(\beta, \alpha) [A(\alpha)]^{-1} A(\alpha, \beta),$$

where $\beta = \langle n \rangle \setminus \alpha$. A well-known result due to Carlson and Markham [1] states that the Schur complements of strictly diagonally dominant matrices are diagonally dominant.

For an $n \times n$ nonnegative and irreducible matrix $A$, Meyer [2,3] introduced the notion of the Perron complement. Again, let $\alpha \subset \langle n \rangle$ and $\beta = \langle n \rangle \setminus \alpha$. Then the Perron complement of $A(\alpha)$ in $A$ is given by

$$P(A/A(\alpha)) = A(\beta) + A(\beta, \alpha) [\rho(A) I - A(\alpha)]^{-1} A(\alpha, \beta),$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. Recall that as $A$ is irreducible, $\rho(A) > \rho(A(\alpha))$, so that the expression on the right-hand side of (3) is well defined, and we observe that $\rho(A) I - A(\alpha)$ is an M-matrix and thus $(\rho(A) I - A(\alpha))^{-1} \geq 0$. Meyer [2,3] has derived several interesting and useful properties of $P(A/A(\alpha))$, such as $P(A/A(\alpha))$ is also nonnegative and irreducible, and $\rho(P(A/A(\alpha))) = \rho(A)$. In addition, the Perron complements of inverse M-matrices [4] have also been studied.

For any $\alpha \subset \langle n \rangle$ and for any $t \geq \rho(A)$, let the extended Perron complement at $t$ be the matrix

$$P_t(A/A(\alpha)) = A(\beta) + A(\beta, \alpha) [t I - A(\alpha)]^{-1} A(\alpha, \beta),$$

which is also well defined since $t \geq \rho(A) > \rho(A(\alpha))$.

In this paper, we shall show, in Section 2, that the Perron complement of a diagonally dominant and nonnegative irreducible matrix $A$, $P(A/A(\alpha)) = A(\beta) + A(\beta, \alpha) [\rho(A) I - A(\alpha)]^{-1} A(\alpha, \beta)$, is diagonally dominant only if $\rho(A) \geq \max_{i \in \alpha} \sum_{j=1}^{n} |a_{ij}|$. In Section 3, we shall show a similar result for H-matrices.

## 2 Perron Complements of Diagonally Dominant Matrices

First recall the following result proved in [2].

**Lemma 2.1** ([2]). If $A$ is a nonnegative irreducible matrix with spectral radius $\rho(A)$, and let $\alpha \subset \langle n \rangle$, $\alpha \neq \phi$ and $\beta = \langle n \rangle \setminus \alpha$. Then the Perron complement

$$P(A/A(\alpha)) = A(\beta) + A(\beta, \alpha) [\rho(A) I - A(\alpha)]^{-1} A(\alpha, \beta)$$

is also a nonnegative irreducible matrix with spectral radius $\rho(A)$.

We are now in a position to state the main result of the paper on the Perron complements of diagonally dominant matrices.
THEOREM 2.2. Let $A$ be an $n \times n$ diagonally dominant and nonnegative irreducible matrix with spectral radius $\rho(A)$, and let $\alpha \subset \langle n \rangle$, $\alpha \neq \phi$ and $\beta = \langle n \rangle \setminus \alpha$. Then, for $\rho(A) \geq \max_{i \in \alpha} \sum_{j=1}^{n} |a_{ij}|$

$$P(A/A(\alpha)) = A(\beta) + A(\beta, \alpha) [\rho(A) I - A(\alpha)]^{-1} A(\alpha, \beta)$$

is a diagonally dominant and nonnegative irreducible matrix.

PROOF. Let $\alpha = \{i_1, i_2, ..., i_k\}$ and $\beta = \{j_1, j_2, ..., j_l\}$, where $k + l = n$. Denote $|A| = (|a_{ij}|)$. Since $A$ is a diagonally dominant matrix, we have, for any $i \in \langle n \rangle$,

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$$

or

$$|a_{j_t,i_s}| \geq \sum_{s=1, \neq t} |a_{j_t,j_s}| + \sum_{s=1}^{k} |a_{j_t,i_s}|,$$  \(\text{(5)}\)

where $j_t, j_s \in \beta, i_s \in \alpha$. Note that $A$ is an irreducible and nonnegative matrix, then $\rho(A) > \rho(A(\alpha))$, so that $\rho(A) I - A(\alpha)$ is an M-matrix. Then we have

$$(\rho(A) I - A(\alpha))^{-1} \geq 0 \text{ and } a_{ij} \geq 0.$$  \(\text{(6)}\)

By $\rho(A) \geq \max_{i \in \alpha} \sum_{j=1}^{n} |a_{ij}|$, we have

$$\rho(A) \geq \max_{i \in \alpha} \left( \sum_{j \in \beta} |a_{ij}| + \sum_{j \in \alpha} |a_{ij}| \right) = \max_{i_v \in \alpha} \sum_{t=1}^{l} a_{i_v,j_t} + \max_{i_v \in \alpha} \sum_{t=1}^{k} a_{i_v,i_t}.$$  \(\text{(7)}\)

If $\max_{i_v \in \alpha} \sum_{t=1}^{l} a_{i_v,j_t} = 0$, then $P(A/A(\alpha)) = A(\beta)$. Thus, the matrix $P(A/A(\alpha))$ is diagonally dominant. If $\max_{i_v \in \alpha} \sum_{t=1}^{l} a_{i_v,j_t} > 0$, then, by (7), we have

$$0 < \max_{i_v \in \alpha} \sum_{t=1}^{l} a_{i_v,j_t} \leq \rho(A) - \max_{i_v \in \alpha} \sum_{t=1}^{k} a_{i_v,i_t}.$$  \(\text{(8)}\)

Thus

$$\max_{i_v \in \alpha} \frac{\sum_{t=1}^{l} a_{i_v,j_t}}{\rho(A) - \sum_{t=1}^{k} a_{i_v,i_t}} \leq 1.$$  \(\text{(9)}\)

Denote

$$x = (\rho(A) I - A(\alpha))^{-1} \left( \sum_{s=1}^{l} a_{i_1,j_s}, ..., \sum_{s=1}^{l} a_{i_k,j_s} \right)^{\dagger}$$  \(\text{(10)}\)
or

\[(\rho (A) I - A(\alpha)) x = \left( \sum_{s=1}^{l} a_{i_1 j_s}, \ldots, \sum_{s=1}^{l} a_{i_k j_s} \right)^\dagger.\]

Letting \(x_v = \max\{x_1, x_2, \ldots, x_k\}\), where \(x_i\) is the \(i\)-th component of \(x\), we obtain

\[
\sum_{s=1}^{l} a_{i_v j_s} = (\rho (A) - a_{i_v i_v}) x_v + \sum_{t=1, \neq v}^{k} (-a_{i_v i_t}) x_t \\
\geq (\rho (A) - a_{i_v i_v}) \sum_{t=1, \neq v}^{k} (-a_{i_v i_t}) x_v \\
= (\rho (A) - \sum_{t=1}^{k} a_{i_v i_t}) x_v.
\]

By (8), we have

\[
x_v \leq \frac{\sum_{t=1}^{l} a_{i_v j_t}}{\rho (A) - \sum_{t=1}^{k} a_{i_v i_t}} \leq \max_{i_v \in \alpha} \frac{\sum_{t=1}^{l} a_{i_v j_t}}{\rho (A) - \sum_{t=1}^{k} a_{i_v i_t}}.
\]

By (9), we have

\[
x_v \leq 1. \tag{11}
\]

Denote the \((t, s)\)-entry of \(P(A/A(\alpha))\) by \((a'_{j_t j_s})\). Then, for \(t = 1, 2, \ldots, l\), we have

\[
\left| a'_{j_t j_s} \right| - \sum_{s=1, \neq t}^{l} \left| a'_{j_t j_s} \right| \\
= a_{j_t j_s} + (a_{j_t i_1}, \ldots, a_{j_t i_k}) (\rho (A) I - A(\alpha))^{-1} \left( \begin{array}{c} a_{i_1 j_t} \\ \vdots \\ a_{i_k j_t} \end{array} \right) \\
- \sum_{s=1, \neq t}^{l} a_{j_t j_s} + (a_{j_t i_1}, \ldots, a_{j_t i_k}) (\rho (A) I - A(\alpha))^{-1} \left( \begin{array}{c} a_{i_1 j_s} \\ \vdots \\ a_{i_k j_s} \end{array} \right),
\]
so that

\[
|a_{j,t}\| - \sum_{s=1, s\neq t}^l |a_{j,s}\| \geq \left| a_{j,t} \right| - (|a_{j,t}|, \ldots, |a_{j,t}|) \left( (\rho(A) I - A(\alpha))^{-1} \right) \begin{pmatrix} |a_{i,t} - t| \\ \vdots \\ |a_{k,t} - t| \end{pmatrix} - \sum_{s=1, s\neq t}^l \left| a_{j,s} \right| + (|a_{j,s}|, \ldots, |a_{j,s}|) \left( (\rho(A) I - A(\alpha))^{-1} \right) \begin{pmatrix} |a_{j,s}| \\ \vdots \\ |a_{j,s}| \end{pmatrix}
\]

\[
= a_{j,t} - \sum_{s=1, s\neq t}^l a_{j,s} - (a_{j,t}, \ldots, a_{j,t}) (\rho(A) I - A(\alpha))^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}
\]

\[
\geq a_{j,t} - \sum_{s=1, s\neq t}^l \left| a_{j,s} \right| - \sum_{s=1}^k \left| a_{j,s} \right|
\]

\[
\geq 0.
\]

It follows that \( P(A/A(\alpha)) \) is a diagonally dominant matrix. By Lemma 2.1, the matrix \( P(A/A(\alpha)) \) is nonnegative irreducible. This completes the proof.

By Theorem 2.2, we have several immediate results about the extended Perron complements and the Perron complements of strictly diagonally dominant matrices.

**COROLLARY 2.3.** Let \( A \) be an \( n \times n \) diagonally dominant and nonnegative irreducible matrix with spectral radius \( \rho(A) \), and let \( \alpha \subset \langle n \rangle, \alpha \neq \emptyset \) and \( \beta = \langle n \rangle \setminus \alpha \). Then, for any \( t \in [\rho(A), \infty) \) and \( \rho(A) \geq \max_{i \in \alpha} \sum_{j=1}^n |a_{ij}| \),

\[
P_t(A/A(\alpha)) = A(\beta) + A(\beta, \alpha) [tI - A(\alpha)]^{-1} A(\alpha, \beta)
\]

is a diagonally dominant and nonnegative irreducible matrix.

**COROLLARY 2.4.** Let \( A \) be an \( n \times n \) strictly diagonally dominant and nonnegative irreducible matrix with spectral radius \( \rho(A) \), and let \( \alpha \subset \langle n \rangle, \alpha \neq \emptyset \) and \( \beta = \langle n \rangle \setminus \alpha \). Then, for any \( t \in [\rho(A), \infty) \) and \( \rho(A) \geq \max_{i \in \alpha} \sum_{j=1}^n |a_{ij}|, P_t(A/A(\alpha)) \) and \( P_t(A/A(\alpha)) \) are strictly diagonally dominant and nonnegative irreducible matrices.
3 Perron Complements of H-matrices

In this section, we obtain a theorem of the Perron complements of H-matrices.

THEOREM 3.1. Let $A$ be an $n \times n$ nonnegative irreducible H-matrix with spectral radius $\rho(A)$, and let $\alpha \subset \langle n \rangle$, $\alpha \neq \emptyset$ and $\beta = \langle n \rangle \setminus \alpha$. Then, for $\rho(A) \geq \max_{i \in \alpha} \sum_{j=1}^{n} |a_{ij}| \geq 2|a_{ii}|, i \in \alpha$,

$$P(A/A(\alpha)) = A(\beta) + A(\beta, \alpha)[\rho(A)I - A(\alpha)]^{-1}A(\alpha, \beta)$$

is a nonnegative irreducible H-matrix.

PROOF. Let $\alpha = \{i_1, i_2, ..., i_k\}$ and $\beta = \{j_1, j_2, ..., j_l\}$, where $k + l = n$. Since $A$ is an H-matrix, then there exists a positive diagonal matrix $X = \text{diag}(x_1, x_2, ..., x_n) > 0$

such that $X^{-1}AX$ is a strictly diagonally dominant matrix, i.e.,

$$|a_{ii}| > \sum_{j \neq i} \frac{x_j}{x_i} |a_{ij}|, i \in \langle n \rangle.$$ 

Suppose that $B = (b_{ij}) = X^{-1}AX$, we have $\rho(B) = \rho(A)$ and $B$ is a strictly diagonally dominant matrix. Since

$$\rho(A) \geq \max_{i \in \alpha} \sum_{j=1}^{n} |a_{ij}| \geq 2|a_{ii}|, i \in \alpha$$

and

$$|a_{ii}| > \sum_{j \neq i} \frac{x_j}{x_i} |a_{ij}|, i \in \langle n \rangle,$$

we have $\rho(B) = \rho(A) \geq \max_{i \in \alpha} \sum_{j=1}^{n} |b_{ij}|$. Then, by Corollary 2.4, $P(B/B(\alpha))$ is a strictly diagonally dominant matrix and

$$B = \begin{bmatrix}
a_{11} & \frac{x_2}{x_1} a_{12} & \cdots & \frac{x_n}{x_1} a_{1n} \\
\frac{x_2}{x_2} a_{21} & a_{22} & \cdots & \frac{x_n}{x_2} a_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{x_n}{x_n} a_{n1} & \frac{x_n}{x_n} a_{n2} & \cdots & a_{nn}
\end{bmatrix}.$$
Let $D = \text{diag}(x_{j_1}, x_{j_2}, \ldots, x_{j_l}) > 0$. Then,

$$
P(\beta / \alpha) = B(\beta) + B(\beta, \alpha) \left[ \rho(B) I - B(\alpha) \right]^{-1} B(\alpha, \beta)
$$

$$
= \begin{bmatrix}
    b_{j_1j_1} & \cdots & b_{j_1j_l} \\
    \vdots & \ddots & \vdots \\
    b_{j_lj_1} & \cdots & b_{j_lj_l}
\end{bmatrix}
+ \begin{bmatrix}
    b_{j_1i_1} & \cdots & b_{j_1i_k} \\
    \vdots & \ddots & \vdots \\
    b_{j_li_1} & \cdots & b_{j_li_k}
\end{bmatrix}
\times \begin{bmatrix}
    \rho(B) - b_{i_1i_1} & \cdots & -b_{i_1i_k} \\
    \vdots & \ddots & \vdots \\
    -b_{i_li_1} & \cdots & \rho(B) - b_{i_li_k}
\end{bmatrix}
\times \begin{bmatrix}
    b_{i_1j_1} & \cdots & b_{i_1j_l} \\
    \vdots & \ddots & \vdots \\
    b_{i_lj_1} & \cdots & b_{i_lj_l}
\end{bmatrix}
= \text{diag}\left(\frac{1}{x_{j_1}}, \ldots, \frac{1}{x_{j_l}}\right)
\begin{bmatrix}
    a_{j_1j_1} & \cdots & a_{j_1j_l} \\
    \vdots & \ddots & \vdots \\
    a_{j_lj_1} & \cdots & a_{j_lj_l}
\end{bmatrix}
\text{diag}(x_{j_1}, \ldots, x_{j_l})
+ \text{diag}\left(\frac{1}{x_{j_1}}, \ldots, \frac{1}{x_{j_l}}\right)
\begin{bmatrix}
    a_{j_1i_1} & \cdots & a_{j_1i_k} \\
    \vdots & \ddots & \vdots \\
    a_{j_li_1} & \cdots & a_{j_li_k}
\end{bmatrix}
\text{diag}(x_{i_1}, \ldots, x_{i_k})
\times \text{diag}\left(\frac{1}{x_{i_1}}, \ldots, \frac{1}{x_{i_k}}\right)
\begin{bmatrix}
    \rho(A) - a_{i_1i_1} & \cdots & -a_{i_1i_k} \\
    \vdots & \ddots & \vdots \\
    -a_{i_li_1} & \cdots & \rho(A) - a_{i_li_k}
\end{bmatrix}
\text{diag}(x_{i_1}, \ldots, x_{i_k})
\times \text{diag}\left(\frac{1}{x_{i_1}}, \ldots, \frac{1}{x_{i_k}}\right)
\begin{bmatrix}
    a_{i_1j_1} & \cdots & a_{i_1j_l} \\
    \vdots & \ddots & \vdots \\
    a_{i_lj_1} & \cdots & a_{i_lj_l}
\end{bmatrix}
\text{diag}(x_{j_1}, \ldots, x_{j_l})
= D^{-1} A(\beta) D + D^{-1} A(\beta, \alpha) \left[ \rho(A) I - A(\alpha) \right]^{-1} A(\alpha, \beta) D
= D^{-1} P(\alpha / \alpha) D.
$$

Note that the matrix

$$
P(\beta / \alpha) = D^{-1} P(\alpha / \alpha) D
$$

is strictly diagonally dominant, then $P(\alpha / \alpha)$ is an H-matrix. By Lemma 2.1, we have the matrix $P(\alpha / \alpha)$ is nonnegative irreducible. This completes the proof.
4 Example

Let
\[
A = \begin{pmatrix}
3 & 1 & 1 & 0 \\
1 & 3 & 0 & 1 \\
2 & 1 & 4 & 1 \\
1 & 2 & 1 & 4
\end{pmatrix}.
\]

Obviously, \(A\) is a diagonally dominant and nonnegative irreducible H-matrix. And,
\[
\rho(A) = 6.3028 \geq \max_{i \in \alpha} \sum_{j=1}^{n} |a_{ij}| \geq 2 |a_{ii}|, \; i \in \alpha, \alpha = \{1\} \text{ or } \{1, 2\}.
\]

Then,
\[
P(A/A(\alpha)) = \begin{pmatrix}
3.3028 & 0.3028 & 1 \\
1.6055 & 4.6055 & 1 \\
2.3028 & 1.3028 & 4
\end{pmatrix},
\]

where \(\alpha = \{1\}\), is a diagonally dominant H-matrix. And,
\[
P(A/A(\alpha)) = \begin{pmatrix}
4.7675 & 1.5351 \\
1.5351 & 4.7675
\end{pmatrix},
\]

where \(\alpha = \{1, 2\}\), is a diagonally dominant H-matrix.

References


