An Answer To The Conjecture Of Satnoianu

Yu Miao, Shou Fang Xu, Ying Xia Chen

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Abstract

In this short paper, we obtain an answer to the conjecture of Satnoianu by a simpler method in the view of probability theory. The conditions of our results are independent with some known answers.

1 Introduction

In [2], Mazur proposed the open problem: if \(a, b, c\) are positive real numbers such that \(abc > 2^9\), then

\[
\frac{1}{\sqrt{1+a}} + \frac{1}{\sqrt{1+b}} + \frac{1}{\sqrt{1+c}} \geq \frac{3}{\sqrt{1+abc}}.
\]  

(1)

In fact, in 2001, Satnoianu [3] has studied the following inequality

\[
\sum_{cyclic} \frac{a}{\sqrt{a^2 + \lambda bc}} \geq \frac{3}{\sqrt{1 + \lambda}} \quad (a, b, c > 0, \lambda \geq 8).
\]  

(2)

In addition, Satnoianu proposed the following inequality as a conjecture

\[
\sum_{i=1}^{n} \left( \frac{x_i^{n-1} + \lambda \prod_{k \neq i} x_k}{x_i^{n-1} + \lambda} \right)^{\frac{1}{n-1}} \geq n(1 + \lambda)^{-\frac{1}{n-1}}.
\]  

(3)

Shortly after the proposed conjecture, Janous [1] gave the proof of the inequality (3) by means of Lagrange’s method of multipliers and Satnoianu [4] obtained a generalized version of inequality (3) as follows

\[
\sum_{i=1}^{n} \left( \frac{x_i^{n-1} + \lambda \prod_{k \neq i} x_k}{\alpha x_i^{n-1} + \beta \prod_{k \neq i} x_k} \right)^{\frac{1}{n-1}} \geq n(\alpha + \beta)^{-\frac{1}{n-1}},
\]  

(4)

*Mathematics Subject Classifications: 26D15
†College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan, 453007, P. R. China. E-mail: yumiao728@yahoo.com.cn
‡Department of mathematics, Xinxiang University, Xinxiang, Henan, 453000, P. R. China
§College of Mathematics and Information Science, Pingdingshan University, Pingdingshan, Henan, 467000, P. R. China
where $n \geq 2$, $x_i > 0$, $i = 1, 2, \ldots, n$, $\alpha, \beta > 0$ and $\beta \geq (n^{n-1} - 1)\alpha$. Recently, Wu [5] established the following more generalized inequality

$$\sum_{i=1}^{n} \left( \frac{x_i^q}{\alpha x_i^q + \beta \prod_{k=1}^{n} x_k^{q/n}} \right)^{\frac{1}{p}} \geq n(\alpha + \beta)^{-\frac{1}{p}},$$

(5)

where $\alpha, \beta, x_i (i = 1, 2, \ldots, n)$ are positive real numbers, $q \in \mathbb{R}$, and $p < 0$, or $p > 0$ with $\beta \geq (n^{\max\{p,1\}} - 1)\alpha$.

If we rewrite the inequality (5) as

$$\frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{\alpha + \beta \exp \left( \frac{1}{n} \sum_{k=1}^{n} \log x_k^q - \log x_i^q \right) } \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}},$$

(6)

then it is easy to see that (6) is equivalent to

$$\mathbb{E} \left( \frac{X^\alpha X + \beta \exp \left\{ \mathbb{E} \log X \right\} }{\alpha X + \beta \exp \left\{ \mathbb{E} \log X \right\} } \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}},$$

(7)

where $X$ is a random variable taking values $x_1^q, x_2^q, \ldots, x_n^q$ with the probability $P(X = x_i^q) = \frac{1}{n}$ and $\mathbb{E}(X)$ denotes the mathematical expectation of $X$. In fact, $X$ can be any positive random variable. Hence we could generalize the conjecture of Satnoianu as: "Under what conditions does the inequality (7) holds?"

2 Main Results

Before our works, we need give the following useful

**LEMMA 1.** Let $f(x) = (a + bx^p)^p$, where $a, b > 0$, $x \in \mathbb{R}$. If $p > 0$ or if $p < 0$ with $pbe^x + a \leq 0$, then $f(x)$ is a convex function.

**PROOF.** The method is elementary. Since a twice differentiable function of one variable is convex on an interval if and only if its second derivative is non-negative and

$$f'(x) = pb(a + bx)^{p-1}e^x,$$

$$f''(x) = p(p-1)b^2(a + bx)^{p-2}e^{2x} + pb(a + bx)^{p-1}e^x = pbe^x(a + bx)^{p-2}[(p-1)be^x + (a + bx)] = pbe^x(a + bx)^{p-2}[pbe^x + a],$$

the desired result is easy to be obtained.

**PROPOSITION 1.** Let random variable $X > 0$ a.e. and $\alpha, \beta > 0$. If $p < 0$ or if $p > 0$ with $X \leq \beta e^{\mathbb{E} \log X / (\alpha p)}$ a.e., then we have

$$\mathbb{E} \left( \frac{X}{\alpha X + \beta \exp \left\{ \mathbb{E} \log X \right\} } \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}.$$
PROOF. Let \( Y = -\log X \), then (8) is equivalent to

\[
E \left( \frac{1}{\alpha + \beta e^{-\alpha e^X}} \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}.
\]  

(9)

By Lemma 1. and Jensen’s inequality, the proof is easy to be obtained.

From the above proposition, we have the following result and the proof is easy.

**THEOREM 1.** Let \( \alpha, \beta > 0 \) and \( X \) be a discrete random variable taking positive numbers \( x_1, x_2, \ldots, x_n \) with \( P(X = x_i) = a_i \), where \( \sum_{i=1}^{n} a_i = 1 \). In addition, let \( M = \max\{x_i, 1 \leq i \leq n\} \) and \( m = \min\{x_i, 1 \leq i \leq n\} \). If \( p < 0 \) or if \( p > 0 \) with \( M/m \leq \beta/((\alpha p) \), then we have

\[
\sum_{i=1}^{n} a_i \left( \frac{x_i}{\alpha x_i + \beta \prod_{k=1}^{n} x_k^{a_k}} \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}.
\]  

(10)

In particular, if \( a_1 = a_2 = \cdots = a_n = \frac{1}{n} \), we have

\[
\sum_{i=1}^{n} \left( \frac{x_i}{\alpha x_i + \beta \prod_{k=1}^{n} x_k^{a_k}} \right)^{\frac{1}{p}} \geq n(\alpha + \beta)^{-\frac{1}{p}}.
\]  

(11)

**REMARK 1.** By comparing the conditions of Theorem 1. with the ones of Wu in [5], we find that these assumptions are independent each other. In fact, the only difference is between “\( M/m \leq \beta/((\alpha p) \)” and “\( \beta \geq (n^{\max\{p,1\}} - 1)\alpha \)” from that we can not judge which condition is weaker than the other.

**REMARK 2.** For the infinite sequence \( \{x_i\}_{i=1}^{\infty} \), let \( \sum_{i=1}^{\infty} a_i = 1, M = \sup_{i \geq 1} x_i < \infty \) and \( m = \inf_{i \geq 1} x_i > 0 \), then by the same discussions as Theorem 1., we have

\[
\sum_{i=1}^{\infty} a_i \left( \frac{x_i}{\alpha x_i + \beta \prod_{k=1}^{\infty} x_k^{a_k}} \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}.
\]  

(12)

The following result is the integral form of the conjecture of Satnoianu.

**THEOREM 2.** Let \( \alpha, \beta > 0 \) and \( X \) be a positive continuous random variable on \((0, \infty)\) with the probability density function \( f(x) \). If \( p < 0 \) or if \( p > 0 \) with \( X \leq \beta e^{E \log X}/(\alpha p) \) a.e., then we have

\[
\int_{0}^{\infty} \left( \frac{x}{\alpha x + \beta \exp \left\{ \int_{0}^{x} \log x f(x) dx \right\} } \right)^{\frac{1}{p}} f(x) dx \geq (\alpha + \beta)^{-\frac{1}{p}}.
\]  

(13)

In particular, if \( X \) possesses uniform distribution on the support interval \([a, b]\), i.e., the probability density function of \( X \) is equal to \((b-a)^{-1}, x \in [a, b] \) and zero elsewhere. Then if \( b/a \leq \beta/(\alpha p) \), then we have

\[
\frac{1}{b-a} \int_{a}^{b} \left( \frac{x}{\alpha x + \beta \exp \left\{ \int_{a}^{x} \log x dx \right\} } \right)^{\frac{1}{p}} dx \geq (\alpha + \beta)^{-\frac{1}{p}}.
\]  

(14)
References


