Some Sharp Simpson Type Inequalities And Applications∗

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Abstract

Some sharp Simpson type inequalities are proved. Applications in numerical integration are also considered.

1 Introduction

Given a real function of a real variable, let us write

\[ f(\alpha|\beta) := f(\alpha) + 4f\left(\frac{\alpha + \beta}{2}\right) + f(\beta). \]

In [1], Ujević proved the following interesting sharp classical Simpson type inequality.

THEOREM 1. Let \( f : [a, b] \rightarrow \mathbb{R} \) be an absolutely continuous function whose derivative \( f' \in L^2(a, b) \). Then

\[ \left| \int_a^b f(x) \, dx - \frac{b-a}{6} f(a|b) \right| \leq \frac{(b-a)^{\frac{3}{2}}}{6} \sqrt{\sigma(f')}, \]  

(1)

where \( \sigma(\cdot) \) is defined by

\[ \sigma(f) = \| f \|_2^2 - \frac{1}{b-a} \left( \int_a^b f(t) \, dt \right)^2 \]  

(2)

and

\[ \| f \|_2 := \left[ \int_a^b f^2(t) \, dt \right]^{\frac{1}{2}}. \]

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Inequality (1) is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.

An application in numerical integration has been given as

**Theorem 2.** Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem 1 hold. Then

$$\left| \int_a^b f(x) \, dx - \frac{h}{6} \sum_{i=0}^{n-1} f(x_i | x_{i+1}) \right| \leq \frac{b-a}{6n} \sigma_n(f) \leq \frac{b-a}{6\sqrt{n}} \omega_n(f),$$

where $\sigma_n(f)$ and $\omega_n(f)$ are defined by

$$\sigma_n(f) = \sum_{i=0}^{n-1} \sqrt{\frac{b-a}{n} \|f'\|^2 - |f(x_{i+1}) - f(x_i)|^2},$$

and

$$\omega_n(f) = [(b-a)\|f'\|^2 - \frac{1}{n}(f(b) - f(a))^2]^{\frac{1}{2}}.$$

Obviously, the inequality (3) seems as if it is complicated and not convenient to obtain the error bounds. Recently in [2] the inequality (3) has been revised and improved as

$$\left| \int_a^b f(x) \, dx - \frac{h}{6} \sum_{i=0}^{n-1} f(x_i | x_{i+1}) \right| \leq \left( \frac{b-a}{6n} \sqrt{\sigma(f')} \right).$$

In this paper, we will further derive some sharp Simpson type inequalities. Applications in numerical integration are also considered.

## 2 Two More Sharp Classical Simpson Type Inequalities

We begin with the following result.

**Theorem 3.** Let $f : [a, b] \rightarrow \mathbb{R}$ be such that $f'$ is absolutely continuous on $[a, b]$ and $f'' \in L_2[a, b]$. Then we have

$$\left| \int_a^b f(x) \, dx - \frac{b-a}{6} f(a) |b| \right| \leq \frac{(b-a)^{\frac{3}{2}}}{12\sqrt{30}} \sqrt{\sigma(f'')}.$$

Inequality (4) is sharp in the sense that the constant $\frac{1}{12\sqrt{30}}$ cannot be replaced by a smaller one.

**Proof.** Let us define the function

$$S_2(x) := \begin{cases} \frac{(x-a)^2}{2} - \frac{(b-a)(x-a)}{6}, & x \in \left[ a, \frac{a+b}{2} \right], \\ \frac{(x-b)^2}{2} + \frac{(b-a)(x-b)}{6}, & x \in \left( \frac{a+b}{2}, b \right). \end{cases}$$

(5)
Integrating by parts, we obtain
\[
\int_a^b S_2(x) f''(x) \, dx = \int_a^b f(x) \, dx - \frac{b-a}{6} f(a|b).
\]  
(6)

By elementary calculus, we have
\[
\int_a^b S_2(x) \, dx = 0, \quad \int_a^b S_2^2(x) \, dx = \frac{(b-a)^5}{4320}.
\]  
(7)

Thus from (6), (7) and (2), we can easily get
\[
\left| \int_a^b f(x) \, dx - \frac{b-a}{6} f(a|b) \right| = \left| \int_a^b S_2(x) f''(x) \, dx \right|
\leq \left( \int_a^b S_2^2(x) \, dx \right)^{\frac{1}{2}} \left\{ \int_a^b \left[ f''(x) - \frac{f'(b) - f'(a)}{b-a} \right]^2 \, dx \right\}^{\frac{1}{2}}
= \left[ \frac{(b-a)^5}{4320} \right]^{\frac{1}{2}} \left\{ \|f''\|_2^2 - \frac{[f'(b) - f'(a)]^2}{b-a} \right\}^{\frac{1}{2}}
= \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{30}} \sqrt{\sigma(f'')}. 
\]

We now suppose that (4) holds with a constant \(C > 0\) as
\[
\left| \int_a^b f(x) \, dx - \frac{b-a}{6} f(a|b) \right| \leq C(b-a)^{\frac{5}{2}} \sqrt{\sigma(f'')}. 
\]  
(8)

We may find a function \(f : [a, b] \to \mathbb{R}\) such that \(f'\) is absolutely continuous on \([a, b]\) as
\[
f'(x) = \begin{cases} 
\frac{(x-a)^2}{6} - \frac{(b-a)(x-a)^2}{12} & \text{if } x \in [a, \frac{a+b}{2}], \\
\frac{(x-b)^2}{6} + \frac{(b-a)(x-b)^2}{12} & \text{if } x \in (\frac{a+b}{2}, b]. 
\end{cases}
\]

It follows that
\[
f''(x) = \begin{cases} 
\frac{(x-a)^2}{2} - \frac{(b-a)(x-a)}{6} & \text{if } x \in [a, \frac{a+b}{2}], \\
\frac{(x-b)^2}{2} + \frac{(b-a)(x-b)}{6} & \text{if } x \in (\frac{a+b}{2}, b]. 
\end{cases}
\]  
(9)

By (5)-(7) and (9), it is not difficult to find that the left-hand side of the inequality (8) becomes
\[
L.H.S.(8) = \frac{(b-a)^5}{4320},
\]  
(10)

and the right-hand side of the inequality (8) is
\[
R.H.S.(8) = \frac{C(b-a)^{\frac{5}{2}}}{12\sqrt{30}}. 
\]  
(11)
From (8), (10) and (11), we find that \( C \geq \frac{1}{12 \sqrt{30}} \), proving that the constant \( \frac{1}{12 \sqrt{30}} \) is the best possible in (4).

**THEOREM 4.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be such that \( f'' \) is absolutely continuous on \([a, b]\) and \( f''' \in L^2[a, b] \). Then we have
\[
\left| \int_a^b f(x) \, dx - \frac{b - a}{6} f(a|b) \right| \leq \frac{(b - a)^{\frac{7}{2}}}{48 \sqrt{105}} \sqrt{\sigma(f''')}.
\]

Inequality (12) is sharp in the sense that the constant \( \frac{1}{48 \sqrt{105}} \) cannot be replaced by a smaller one.

**PROOF.** Let us define the function
\[
S_3(x) := \begin{cases} \frac{(x-a)^3}{6} - \frac{(b-a)(x-a)^2}{12} & x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^3}{6} + \frac{(b-a)(x-b)^2}{12} & x \in (\frac{a+b}{2}, b]. \end{cases}
\]

Integrating by parts, we obtain
\[
\int_a^b S_3(x) f'''(x) \, dx = \frac{b - a}{6} f(a|b) - \int_a^b f(x) \, dx.
\]

By elementary calculus, we have
\[
\int_a^b S_3(x) \, dx = 0, \quad \int_a^b S_3^2(x) \, dx = \frac{(b - a)^7}{241920}.
\]

Thus from (14), (15) and (2), we can easily get
\[
\left| \int_a^b f(x) \, dx - \frac{b - a}{6} f(a|b) \right| = \left| \int_a^b S_3(x) f'''(x) \, dx \right|
\leq \left( \int_a^b S_3^2(x) \, dx \right)^{\frac{1}{2}} \left( \int_a^b \left[ f'''(x) - \frac{f''(b) - f''(a)}{b - a} \right]^2 \, dx \right)^{\frac{1}{2}}
\leq \frac{(b - a)^{\frac{7}{2}}}{241920} \left\{ \left\| f''' \right\|_2^2 - \frac{\| f''(b) - f''(a) \|_2^2}{b - a} \right\}^{\frac{1}{2}}
= \frac{(b - a)^{\frac{7}{2}}}{48 \sqrt{105}} \sqrt{\sigma(f''')}.
\]

We now suppose that (12) holds with a constant \( C > 0 \) as
\[
\left| \int_a^b f(x) \, dx - \frac{b - a}{6} f(a|b) \right| \leq C(b - a)^{\frac{7}{2}} \sqrt{\sigma(f''')}.
\]
We may find a function \( f : [a, b] \to \mathbb{R} \) such that \( f'' \) is absolutely continuous on \([a, b]\) as

\[
f''(x) = \begin{cases} 
\frac{(x-a)^3}{24} - \frac{(b-a)(x-a)^3}{36} & \text{if } x \in [a, \frac{a+b}{2}], \\
\frac{(x-a)^3}{24} + \frac{(b-a)(x-b)^3}{36} & \text{if } x \in \left(\frac{a+b}{2}, b\right].
\end{cases}
\]

It follows that

\[
f'''(x) = \begin{cases} 
\frac{(x-a)^3}{6} - \frac{(b-a)(x-a)^2}{12} & \text{if } x \in [a, \frac{a+b}{2}], \\
\frac{(x-a)^3}{6} + \frac{(b-a)(x-b)^2}{12} & \text{if } x \in \left(\frac{a+b}{2}, b\right].
\end{cases}
\]

By (13)-(15) and (17), it is not difficult to find that the left-hand side of the inequality (16) becomes

\[
\text{L.H.S.}(16) = \frac{(b-a)^7}{241920}, \tag{18}
\]

and the right-hand side of the inequality (16) is

\[
\text{R.H.S.}(16) = \frac{C(b-a)^7}{48\sqrt{105}}. \tag{19}
\]

From (16), (18) and (19), we find that \( C \geq \frac{1}{48\sqrt{105}} \), proving that the constant \( \frac{1}{48\sqrt{105}} \) is the best possible in (12).

**Remark 1.** It should be noticed that the classical Simpson type inequalities (1), (4) and (12) have been appeared in [3] without the proofs of their sharpness but with some misprints.

### 3 Two Sharp Generalized Simpson Type Inequalities

In [4], we may find the identity

\[
(-1)^n \int_a^b S_n(x) f^{(n)}(x) \, dx = \int_a^b f(x) dx - \frac{b-a}{6} f(a|x|) + \sum_{k=2}^{\left[ \frac{n-1}{2} \right]} (k-1)(b-a)^{2k+1} \frac{3(2k+1)!}{3^{2k+1}2^{2k-1}} f^{(2k)} \left( \frac{a+b}{2} \right), \tag{20}
\]

where \( \left[ \frac{n-1}{2} \right] \) denotes the integer part of \( \frac{n-1}{2} \) and \( S_n(x) \) is the kernel given by

\[
S_n(x) = \begin{cases} 
\frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^n}{6(n-1)!} & \text{if } x \in [a, \frac{a+b}{2}], \\
\frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^n}{6(n-1)!} & \text{if } x \in \left(\frac{a+b}{2}, b\right].
\end{cases}
\]

By elementary calculus, it is not difficult to get

\[
\int_a^b S_n(x) \, dx = \begin{cases} 
0, & \text{if } n \text{ odd}, \\
\frac{-(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & \text{if } n \text{ even.} \tag{22}
\end{cases}
\]
and
\[
\int_a^b S_n^2(x) \, dx = \frac{(2n^3 - 11n^2 + 18n - 6)(b - a)^{2n+1}}{9(4n^2 - 1)(n!)^2 2^{2n}}. \quad (23)
\]

**THEOREM 5.** Let \( f : [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)} \) is absolutely continuous on \([a, b]\) and \( f^{(n)} \in L^2[a, b] \) where \( n \) is an odd integer. Then we have
\[
\left| \int_a^b f(x) \, dx - \frac{b - a}{6} f(a) + \frac{(n-1)(b - a)^{2k+1}}{3(2k + 1)! 2^{2k-1}} f^{(2k)} \left( \frac{a + b}{2} \right) \right| \leq \frac{1}{3} \frac{(b - a)^{n+\frac{3}{2}}}{2^n n!} \sqrt{\frac{2n^3 - 11n^2 + 18n - 6}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})}. \quad (24)
\]

Inequality (24) is sharp in the sense that the constant \( \frac{1}{3} \frac{1}{2^n n!} \sqrt{\frac{2n^3 - 11n^2 + 18n - 6}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})} \) cannot be replaced by a smaller one.

**PROOF.** From (20), (22), (23) and (2), we can easily get
\[
\left| \int_a^b f(x) \, dx - \frac{b - a}{6} f(a) + \frac{(n-1)(b - a)^{2k+1}}{3(2k + 1)! 2^{2k-1}} f^{(2k)} \left( \frac{a + b}{2} \right) \right| = \left| \int_a^b S_n(x) f^{(n)}(x) \, dx \right|
\]
\[
= \left| \int_a^b S_n(x) \left[ f^{(n)}(x) - \frac{1}{b - a} \int_a^b f^{(n)}(t) \, dt \right] \, dx \right|
\]
\[
\leq \left( \int_a^b S_n^2(x) \, dx \right)^{\frac{1}{2}} \left( \int_a^b \left[ f^{(n)}(x) - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a} \right]^2 \, dx \right)^{\frac{1}{2}}
\]
\[
= \left( \frac{(2n^3 - 11n^2 + 18n - 6)(b - a)^{2n+1}}{9(4n^2 - 1)(n!)^2 2^{2n}} \right)^{\frac{1}{4}} \left( \|f^{(n)}\|_2^2 \left( \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b - a} \right)^2 \right)^{\frac{1}{4}}
\]
\[
= \frac{1}{3} \frac{(b - a)^{n+\frac{3}{2}}}{2^n n!} \sqrt{\frac{2n^3 - 11n^2 + 18n - 6}{4n^2 - 1}} \sqrt{\sigma(f^{(n)})}.
\]

We now suppose that (24) holds with a constant \( C > 0 \) as
\[
\left| \int_a^b f(x) \, dx - \frac{b - a}{6} f(a) + \frac{(n-1)(b - a)^{2k+1}}{3(2k + 1)! 2^{2k-1}} f^{(2k)} \left( \frac{a + b}{2} \right) \right| \leq C(b - a)^{n+\frac{3}{2}} \sqrt{\sigma(f^{(n)})}. \quad (25)
\]

We may find a function \( f : [a, b] \to \mathbb{R} \) such that \( f^{(n-1)} \) is absolutely continuous on \([a, b]\) as
\[
f^{(n-1)}(x) = \begin{cases} 
    \frac{(x-a)^{n+1}}{(n+1)!} & \text{if } x \in [a, \frac{a+b}{2}] \\
    \frac{(a-b)^{n+1}}{(n+1)!} & \text{if } x \in (\frac{a+b}{2}, b].
\end{cases}
\]
It follows that

$$f^{(n)}(x) = \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{6(n-1)!} & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{6(n-1)!} & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases} \quad (26)$$

By (20)-(23) and (26), it is not difficult to find that the left-hand side of the inequality (25) becomes

$$L.H.S.(25) = \frac{(2n^3 - 11n^2 + 18n - 6)(b - a)^{2n+1}}{9(4n^2 - 1)(n!)^22^{2n}}, \quad (27)$$

and the right-hand side of the inequality (25) is

$$R.H.S.(25) = \frac{1}{3} \frac{1}{2^n n!} \sqrt{2n^3 - 11n^2 + 18n - 6} \frac{C(b - a)^{2n+1}}{4n^2 - 1}. \quad (28)$$

From (25), (27) and (28), we find that

$$C \geq \frac{1}{3} \frac{1}{2^n n!} \sqrt{2n^3 - 11n^2 + 18n - 6} \frac{C(b - a)^{2n+1}}{4n^2 - 1},$$

proving that the constant \(\frac{1}{3} \frac{1}{2^n n!} \sqrt{2n^3 - 11n^2 + 18n - 6} \) is the best possible in (24).

**REMARK 2.** It is clear that Theorem 1 and Theorem 4 can be regarded as special cases of Theorem 5.

**THEOREM 6.** Let \(f : [a, b] \to \mathbb{R}\) be such that \(f^{(n-1)}\) is absolutely continuous on \([a, b]\) and \(f^{(n)} \in L^2[a, b]\) where \(n\) is an even integer. Then we have

$$\left| \int_a^b f(x) \, dx - \frac{b - a}{6} f(a|b) + \sum_{k=2}^{n+1} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}(\frac{a+b}{2}) \right|$$

$$\leq \frac{1}{3} \frac{1}{2^n (n+1)!} \sqrt{2n^3 - 11n^2 + 14n^3 + 4n^2 + 2n - 2} \sqrt{\sigma(f^{(n)})}. \quad (29)$$

Inequality (29) is sharp in the sense that the constant \(\frac{1}{3} \frac{1}{2^n (n+1)!} \sqrt{2n^3 - 11n^2 + 14n^3 + 4n^2 + 2n - 2} \) cannot be replaced by a smaller one.
We may find a function $C > 0$ as

$$f(x) = \sum_{k=2}^{\infty} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)} \left( \frac{a+b}{2} \right) \left( \frac{1}{2} \right) + \frac{(n-2)(b-a)^n}{3(n+1)!2^n} |f^{(n-1)}(b) - f^{(n-1)}(a)|$$

We now suppose that (29) holds with a constant $C > 0$ as

$$\left| \int_a^b f(x) \, dx - \frac{b-a}{6} f(a) + \sum_{k=2}^{\infty} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)} \left( \frac{a+b}{2} \right) \right| \leq \frac{(n-2)(b-a)^n}{3(n+1)!2^n} |f^{(n-1)}(b) - f^{(n-1)}(a)|$$

We may find a function $f : [a, b] \to \mathbb{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a, b]$ as

$$f^{(n-1)}(x) = \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!} + \frac{(b-a)(x-a)^n}{6n} + \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^{2n+1}} & \text{if } x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^{n+1}}{(n+1)!} + \frac{(b-a)(x-b)^n}{6n} - \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^{2n+1}} & \text{if } x \in (\frac{a+b}{2}, b]. \end{cases}$$
It follows that
\[
   f^{(n)}(x) = \begin{cases} 
   \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{6(n-1)!} & \text{if } x \in [a, \frac{a+b}{2}], \\
   \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{6(n-1)!} & \text{if } x \in \left(\frac{a+b}{2}, b\right]. 
   \end{cases} 
\] 
(31)

By (20)-(23) and (31), it is not difficult to find that the left-hand side of the inequality (30) becomes
\[
   L.H.S.(30) = \frac{2n^5 - 11n^4 + 14n^3 + 4n^2 + 2n - 2}{9(4n^2 - 1)(n + 1)!^{22n}}. 
\] 
(32)

and the right-hand side of the inequality (30) is
\[
   R.H.S.(30) = \frac{1}{3} \frac{1}{2^{n(n+1)!}} \sqrt{\frac{2n^5 - 11n^4 + 14n^3 + 4n^2 + 2n - 2}{4n^2 - 1}} C(b - a)^{2n+1}. 
\] 
(33)

From (30), (32) and (33), we find that \( C \geq \frac{1}{3} \frac{1}{2^{n(n+1)!}} \sqrt{\frac{2n^5 - 11n^4 + 14n^3 + 4n^2 + 2n - 2}{4n^2 - 1}} \),
proving that the constant \( \frac{1}{3} \frac{1}{2^{n(n+1)!}} \sqrt{\frac{2n^5 - 11n^4 + 14n^3 + 4n^2 + 2n - 2}{4n^2 - 1}} \) is the best possible in (29).

REMARK 3. It is clear that Theorem 3 can be regarded as a special case of Theorem 6.

REMARK 4. If we take \( n = 4 \) in Theorem 6, we get a sharp perturbed Simpson type inequality as
\[
   \left| \int_a^b f(t) \, dt - \frac{1}{b-a} f(a) + \frac{(b-a)^4}{2880} [f(3)(b) - f(3)(a)] \right| \leq \frac{1}{2880} \sqrt{\frac{11}{14} (b-a)^\frac{7}{2} \sqrt{\sigma(f^{(4)})}}. 
\] 
(34)

Also, it should be noticed that inequality (34) has been appeared in [3] without a proof of its sharpness but with a misprint.

4 Applications in Numerical Integration

We restrict further considerations to the applications of Theorem 3 and Theorem 4.

THEOREM 7. Let \( \pi = \{x_0 = a < x_1 < \cdots < x_n = b\} \) be a given subdivision of the interval \([a, b]\) such that \( h_i = x_{i+1} - x_i = h = \frac{b-a}{n} \) and let the assumptions of Theorem 3 hold. Then we have
\[
   \left| \int_a^b f(x) \, dx - \frac{h}{6} \sum_{i=0}^{n-1} f(x_i|x_{i+1}) \right| \leq \frac{(b-a)^{\frac{7}{2}}}{12\sqrt{30n^2}} \sqrt{\sigma(f^{(7)})}. 
\] 
(35)

PROOF. From (4) in Theorem 3 we obtain
\[
   \left| \int_{x_i}^{x_{i+1}} f(t) \, dt - \frac{h}{6} f(x_i|x_{i+1}) \right| \leq \frac{h^{\frac{7}{2}}}{12\sqrt{30}} \left\{ \int_{x_i}^{x_{i+1}} [f''(t)]^2 \, dt - \frac{1}{h} [f'(x_{i+1}) - f'(x_i)]^2 \right\}^{\frac{7}{4}}. 
\] 
(36)
By summing (36) over $i$ from 0 to $n - 1$ and using the generalized triangle inequality, we get

$$\left| \int_a^b f(t) \, dt - \frac{h}{6} \sum_{i=0}^{n-1} f(x_i \mid x_{i+1}) \right| \leq \frac{h^\frac{7}{2}}{12\sqrt{30}} \sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} \left[ f''(t) \right]^2 \, dt - \frac{1}{h} [f'(x_{i+1}) - f'(x_i)]^2 \right\}^{\frac{1}{2}}. \quad (37)$$

By using the Cauchy inequality twice, it is not difficult to obtain

$$\sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} \left[ f''(t) \right]^2 \, dt - \frac{1}{h} [f'(x_{i+1}) - f'(x_i)]^2 \right\}^{\frac{1}{2}} \leq \sqrt{\pi} \left\{ \int_a^b \left[ f''(t) \right]^2 \, dt - \frac{n}{b-a} \sum_{i=0}^{n-1} [f'(x_{i+1}) - f'(x_i)]^2 \right\}^{\frac{1}{2}} \leq \sqrt{\pi} \left\{ ||f''||_2^2 - \frac{\left[ f'(b) - f'(a) \right]^2}{b-a} \right\}^{\frac{1}{2}}. \quad (38)$$

Consequently, the inequality (35) follows from (37) and (38).

**THEOREM 8.** Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem 4 hold. Then we have

$$\left| \int_a^b f(x) \, dx - \frac{h}{6} \sum_{i=0}^{n-1} f(x_i \mid x_{i+1}) \right| \leq \frac{(b-a)^{\frac{7}{2}}}{48\sqrt{105}n^3} \sqrt{\sigma(f''')} \cdot (39)$$

**PROOF.** From (12) in Theorem 4 we obtain

$$\left| \int_{x_i}^{x_{i+1}} f(t) \, dt - \frac{h}{6} f(x_i \mid x_{i+1}) \right| \leq \frac{h^\frac{7}{2}}{48\sqrt{105}} \left\{ \int_{x_i}^{x_{i+1}} \left[ f''(t) \right]^2 \, dt - \frac{1}{h} [f''(x_{i+1}) - f''(x_i)]^2 \right\}^{\frac{1}{2}}. \quad (40)$$

By summing (40) over $i$ from 0 to $n - 1$ and using the generalized triangle inequality, we get

$$\left| \int_a^b f(t) \, dt - \frac{h}{6} \sum_{i=0}^{n-1} f(x_i \mid x_{i+1}) \right| \leq \frac{h^\frac{7}{2}}{48\sqrt{105}} \sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} \left[ f''(t) \right]^2 \, dt - \frac{1}{h} [f''(x_{i+1}) - f''(x_i)]^2 \right\}^{\frac{1}{2}}. \quad (41)$$
By using the Cauchy inequality twice, it is not difficult to obtain

\[
\sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} \left[ f'''(t) \right]^2 dt - \frac{1}{b-a} \sum_{i=0}^{n-1} \left[ f''(x_{i+1}) - f''(x_i) \right]^2 \right\}^{\frac{1}{2}} \\
\leq \sqrt{n} \left\{ \int_a^b \left[ f'''(t) \right]^2 dt - \frac{n}{b-a} \sum_{i=0}^{n-1} \left[ f''(x_{i+1}) - f''(x_i) \right]^2 \right\}^{\frac{1}{2}}
\]

\leq \sqrt{n} \left\{ \int_a^b \left[ f'''(t) \right]^2 dt - \frac{\left[ f''(b) - f''(a) \right]^2}{b-a} \right\}^{\frac{1}{2}}. \tag{42}
\]

Consequently, the inequality (39) follows from (41) and (42).

References


