Some Discrete Representations Of $q$-Classical Linear Forms

Olfa Fériel Kamech†, Manoubi Mejri‡

Received 25 September 2007

Abstract

We give a discrete measure for some $H_q$-classical forms and some consequent summation formulas.

1 Introduction and Preliminaries

In [4], $H_q$-classical orthogonal polynomials are exhaustively described and integral or discrete representations of corresponding regular forms are given, except in some cases where the problem remains open (see also [3] for the $H_q$-semiclassical case). So, the aim of this contribution is to establish discrete representations of two canonical situations in [4] which are the $q$-analogous of Hermite (for $0 < q < 1$, $q > 1$) and the $q$-analogous of Laguerre (for $q > 1$).

Let $P$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $P'$ be its dual. We denote by $(u, f)$ the action of $u \in P'$ on $f \in P$. In particular, for any $f \in P$, we let $fu$, be the form defined by duality $\langle fu, p \rangle := \langle u, fp \rangle$, $p \in P$.

Let $(\delta_c, p) = p(c)$, $c \in \mathbb{C}$, $p \in P$.

The form $u$ is called regular if we can associate with it a sequence $\{P_n\}_{n \geq 0}$ of monic polynomials, $\deg P_n = n$, $n \geq 0$ such that

$$\langle u, P_m P_n \rangle = r_n \delta_{n,m}, \quad n, m \geq 0; \quad r_n \neq 0, \quad n \geq 0.$$

The sequence $\{P_n\}_{n \geq 0}$ is orthogonal with respect to $u$ and fulfils the standard recurrence relation:

$$\begin{cases}
P_0(x) = 1, & P_1(x) = x - \beta_0, \\
P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), & n \geq 0
\end{cases}$$

\[\tag{1}\]

\[\text{Mathematics Subject Classifications: 42C05, 33C45.}\]

\[\text{†Department of Mathematics, Institut Preparatoire aux Etudes d’Ingenieurs EL Manar 2990 EL Manar, B.P 244 Tunis Tunisia}\]

\[\text{‡Department of Mathematics, Institut Superieur Des Sciences Appliquees et de Technologie Rue Omar Ibn EL Khattab Gabes 6072, Tunisia. e-mail: mejri_manoubi@yahoo.fr}\]

34
with $\gamma_{n+1} \neq 0$, $n \geq 0$.

The form $u$ is said to be normalized if $(u)_0 = 1$ where in general $(u)_n = \langle u, x^n \rangle$, $n \geq 0$, are the moments of $u$. In this paper we suppose that any form will be normalized.

Let us introduce the Hahn’s operator

$$(H_q f)(x) := \frac{f(qx) - f(x)}{(q - 1)x}, \ f \in \mathcal{P}, \ q \in \tilde{\mathbb{C}},$$

where $\tilde{\mathbb{C}} := \mathbb{C} - \left( \{0\} \cup \bigcup_{n \geq 0} \{ z \in \mathbb{C}, z^n = 1 \} \right)$.

By duality we have

$$\langle H_q u, f \rangle = -\langle u, H_q f \rangle, \ u \in \mathcal{P}', \ f \in \mathcal{P}.$$ 

DEFINITION. A form $u$ is called $H_q$-classical when it is regular and there exists two polynomials $\phi$ (monic) and $\psi$ with $\text{deg}(\phi) \leq 2$, $\text{deg}(\psi) = 1$ such that

$$H_q(\phi u) + \psi u = 0.$$  (2)

The corresponding orthogonal sequence $\{P_n\}_{n \geq 0}$ is called $H_q$-classical.

We are going to use the following notations and results $[1,2,5]$

$$(a; q)_n = \begin{cases} 1, & n = 0, \\ \frac{n-1}{\prod_{k=0}^{n-1} (1 - aq^k)}, & n \geq 1. \end{cases}$$  (3)

$$(a; q)_n = (-1)^n a^n q^{\frac{n(n+1)}{2}} (a^{-1}; q^{-1})_n, \ n \geq 0, \ a, q \neq 0.$$  (4)

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \ |q| < 1.$$  (5)

$$(a; q)_n = \begin{cases} (a; q)_\infty, & |q| < 1, \\ \frac{(aq^n; q)_\infty}{(a^{-1}; q^{-1})_\infty}, & |q| > 1. \end{cases}$$  (6)

$$(z; q)_\infty = \sum_{k=0}^{\infty} (-1)^k q^{\frac{k(k-1)}{2}} z^k, \ |q| < 1.$$  (7)

$$\frac{1}{(z, q)_\infty} = \sum_{k=0}^{\infty} \frac{1}{(q; q)_k} z^k, \ |q| < 1, \ |z| < 1.$$  (8)
2 Discrete measure for some $H_q$-classical forms

2.1

Consider the symmetric $H_q$-classical linear form $u$ which is the $q$-analog of Hermite functional. We have [4]

\[
\begin{aligned}
\beta_n &= 0, \ n \geq 0, \\
\gamma_{n+1} &= \frac{1 - q^{n+1}}{2(1 - q)} q^n, \ n \geq 0, \\
H_q(u) + 2 xu &= 0.
\end{aligned}
\tag{9}
\]

\[
\langle u, f \rangle = \left\{ \begin{array}{ll}
\sqrt{2} \pi (q - 1)^{1/2} \langle q^{-2}; q^{-2} \rangle_{\infty} \int_{-\infty}^{+\infty} f(x) \frac{dx}{-2(q - 1)x^2; q^{-2}}_{\infty}, & f \in \mathcal{P}, \ q > 1, \\
K_1 \int_{-\frac{1}{\sqrt{2}(1 - q)^q}}^{\frac{1}{\sqrt{2}(1 - q)^q}} (2q^2(1 - q)x^2; q^2)_{\infty} f(x) dx, & f \in \mathcal{P}, \ 0 < q < 1,
\end{array} \right.
\tag{10}
\]

with

\[
K_1 = \frac{1}{2} \left( \int_{0}^{+\frac{1}{\sqrt{2}(1 - q)^q}} (2q^2(1 - q)x^2; q^2)_{\infty} dx \right)^{-1}.
\tag{11}
\]

\[
(u)_{2n} = 1 \frac{(q; q^2)^n}{2^n (1 - q)^n}, \ (u)_{2n+1} = 0, \ n \geq 0.
\tag{12}
\]

**PROPOSITION 1.** We have the following discrete representations:

For $f \in \mathcal{P}, \ q > 1$

\[
\langle u, f \rangle = \frac{1}{2(q^{-1}; q^{-2})_{\infty}} \sum_{k=0}^{+\infty} (-1)^k q^{-k^2} \left\{ f\left( \frac{-iq^k}{\sqrt{2(q - 1)}} \right) + f\left( \frac{iq^k}{\sqrt{2(q - 1)}} \right) \right\}.
\tag{13}
\]

For $f \in \mathcal{P}, \ 0 < q < 1$

\[
\langle u, f \rangle = 2^{-1}(q; q^2)_{\infty} \sum_{k=0}^{+\infty} \frac{q^k}{(q^2; q^2)_k} \left\{ f\left( \frac{-q^k}{\sqrt{2(1 - q)}} \right) + f\left( \frac{q^k}{\sqrt{2(1 - q)}} \right) \right\}.
\tag{14}
\]

**PROOF.** Let $q > 1$ by (6), equation (12) becomes

\[
(u)_{2n} = \frac{1}{2^n (1 - q)^n} \frac{(q^{2n-1}; q^{-2})_{\infty}}{(q^{-1}; q^{-2})_{\infty}}, \ n \geq 0.
\]

On account of (7), we get

\[
(u)_{2n} = \frac{1}{(q^{-1}; q^{-2})_{\infty}} \sum_{k=0}^{+\infty} (-1)^k q^{-k^2} \left( \frac{iq^k}{\sqrt{2(q - 1)}} \right)^{2n}, \ n \geq 0.
\]
Therefore
\[(u)_{2n} = \left\langle \frac{1}{(q^{-1};q^{-2})_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k}, x^{2n} \right\rangle, n \geq 0.\]

But \((u)_{2n+1} = 0, n \geq 0\), yields to
\[
(u)_n = \langle u, x^n \rangle = \left\langle \frac{1}{2(q^{-1};q^{-2})_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k} \left\{ \delta \frac{-iq^k}{\sqrt{2(q-1)}} + \delta \frac{iq^k}{\sqrt{2(q-1)}} \right\}, x^n \right\rangle, n \geq 0.
\]

Consequently
\[
u = \frac{1}{2(q^{-1};q^{-2})_\infty} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{-k^2}}{(q^{-2};q^{-2})_k} \left\{ \delta \frac{-iq^k}{\sqrt{2(q-1)}} + \delta \frac{iq^k}{\sqrt{2(q-1)}} \right\}.
\]

Then we get the desired result (13).

When \(0 < q < 1\), by virtue of (6), equation (12) becomes
\[
(u)_{2n} = \frac{(q; q^2)_\infty}{2n(1 - q)^n(q^{2n+1}; q^2)_\infty}, n \geq 0,
\]
on account of (8), it follows that
\[
(u)_{2n} = (q; q^2)_\infty \sum_{k=0}^{+\infty} \frac{q^k}{(q^2; q^2)_k} \left\{ \frac{q^k}{\sqrt{2(1-q)}} \right\}^{2n}, n \geq 0.
\]

Then
\[
(u)_n = \left\langle 2^{-1}(q; q^2)_\infty \sum_{k=0}^{+\infty} \frac{q^k}{(q^2; q^2)_k} \left\{ \delta \frac{-iq^k}{\sqrt{2(1-q)}} + \delta \frac{iq^k}{\sqrt{2(1-q)}} \right\}, x^n \right\rangle, n \geq 0.
\]

Consequently, we are lead to
\[
u = 2^{-1}(q; q^2)_\infty \sum_{k=0}^{+\infty} \frac{q^k}{(q^2; q^2)_k} \left\{ \delta \frac{-iq^k}{\sqrt{2(1-q)}} + \delta \frac{iq^k}{\sqrt{2(1-q)}} \right\}.
\]

Hence (14).

### 2.2

Consider the q-analogous of Laguerre linear form \(u\) given in [4,p 68]. We have
\[
\begin{align*}
\beta_n &= \{1 - (1 + q)q^n\}q^{n-1}, n \geq 0, \\
\gamma_{n+1} &= (q^{n+1} - 1)q^{3n}, n \geq 0, \\
H_q(xu) - (q - 1)^{-1}(x + 1)u &= 0. 
\end{align*}
\]
For $q > 1$, we have the following representations [4]:

$$
\langle u, f \rangle = \begin{cases} 
(2\pi \ln q)^{-1/2} q^{-1/8} \int_{-\infty}^{0} |x|^{-3/2} \exp \left( -\frac{\ln^2 |x|}{2 \ln q} \right) f(x) \, dx, \; f \in P, \\
\sum_{k=0}^{+\infty} (-1)^k q^{-k^2} s(k) f(-q^k), \; f \in P,
\end{cases}
\tag{17}
$$

where

$$s(k) = \sum_{m=0}^{+\infty} q^{-\left(\frac{1}{2}m(m+1)+km\right)} \frac{(q^{-1}; q^{-1})_m}{(q^{-1}; q^{-1})_m} (u)_m^{\phi}; \; k \geq 0,
\tag{18}
$$

and $(u)_{2n} = (q-1)^n$, $(u)_{2n+1} = 0$, $n \geq 0$.

The moments of $u$ are given by the following formulas:

$$(u)_n = (-1)^n q^{\frac{1}{2}n(n-1)}, \; n \geq 0.
\tag{19}$$

PROPOSITION 2. The form $u$ possesses the following discrete representation:

For $f \in P$, $q > 1$

$$(-1; q^{-1})_\infty (-q^{-1}; q^{-1})_\infty \langle u, f \rangle = \sum_{k=0}^{+\infty} q^{-\frac{k(k-1)}{2}} \sum_{\mu=0}^{k} q^{-\mu^2+(k-1)\mu} \frac{(q^{-1}; q^{-1})_{\mu}}{(q^{-1}; q^{-1})_{\mu}} f(-q^{2n-k}),
\tag{20}
$$

PROOF. From (4), for (19) we obtain

$$
(u)_n = (-1)^n \frac{(-1; q^n}_{(-1; q^{-1})_n}, \; n \geq 0.
\tag{21}
$$

Let $q > 1$, taking (6) into account, equation (21) can be written in the following way

$$(u)_n = \frac{(-1)^n}{(-1; q^{-1})_\infty (-q^{-1}; q^{-1})_\infty (-q^{-1}; q^{-1})_\infty (-q^{-n}; q^{-1})_\infty}, \; n \geq 0.
\tag{22}
$$

In accordance of (7), we get

$$(u)_n = \frac{(-1)^n}{(-1; q^{-1})_\infty (-q^{-1}; q^{-1})_\infty} \sum_{k=0}^{+\infty} q^{-\frac{k(k-1)}{2}} \frac{(q^{-1}; q^{-1})_k}{(q^{-1}; q^{-1})_k} q^{k(n-1)} \sum_{k=0}^{+\infty} q^{-\frac{k(k-1)}{2}} q^{-kn}, \; n \geq 0.
\tag{23}
$$

Using the Cauchy product, the last expression becomes (for $n \geq 0$)

$$(u)_n = \frac{1}{(-1; q^{-1})_\infty (-q^{-1}; q^{-1})_\infty} \sum_{k=0}^{+\infty} q^{-\frac{k(k-1)}{2}} \sum_{\mu=0}^{k} q^{-\mu^2+(k-1)\mu} \frac{(q^{-1}; q^{-1})_{\mu}}{(q^{-1}; q^{-1})_{\mu}} (q^{-2\mu-k})^n.
\tag{24}
$$

Then, the discrete measure in (20) is deduced.

Acknowledgment. We would like to thank the referee for his valuable review.
References


