Homework #5

(1) Explore shock capturing schemes

Consider the following upwind scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2}\frac{(u_j^n)^2 - (u_{j-1}^n)^2}{\Delta x} = 0$$

for Burgers equation: $u_t + uu_x = 0$, or in conservative form $u_t + \frac{1}{2}(u^2)_x = 0$. The initial data is taken to be $u(x,0) = 1 + \sin(x)$ and the boundary condition is taken to be periodic: $u(0,t) = u(2\pi, t)$

(a) Write a code to compute the solution at time $t = 5$. What happens if you use central difference schemes or down wind scheme instead. Derive the modified equation and explain why the upwind scheme works and the down wind scheme does not work.

(b) When the initial data is taken to be $u(x,0) = -1 + \sin(x)$, what is the upwind scheme?

(c) Can you find a good upwind scheme when the initial data is taken to be $u(x,0) = \sin(x)$?

(2) Explore modern shock capturing scheme and gas dynamics.

The 1d gas dynamics is given by

$$\partial_t U + \partial_x F(U) = 0, \quad U = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \quad F = \begin{bmatrix} m \\ \rho u^2 + p \\ u(E+p) \end{bmatrix}$$

where ρ, u, $m = \rho u$ and E are density, velocity, momentum and total energy, respectively, and the pressure is obtained from the equation of state, $p = (\gamma - 1)(E - \rho u^2/2)$, $\gamma = 1.4$. The equation in multidimensional space is given similarly.

Modern shock capturing scheme are given in a conservative form

$$U_j^{n+1} = U_j^n - \lambda(\hat{F}_{j+\frac{1}{2}} - \hat{F}_{j-\frac{1}{2}}), \quad \lambda = \Delta t/\Delta x$$

where Δt and Δx are temporal and spatial grid sizes, respectively; U_j^n is the approximation to the cell average over the interval $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ and $\hat{F}_{j+\frac{1}{2}}$ is a numerical flux.

The second order nonlinear interpolation for a general data set $\{g_j\}$ on each cell $(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ is given by

$$g_j(x) = g_j + s_j(x - x_j), \quad s_j = \text{minmod}\left(\frac{g_{j+1} - g_j}{\Delta x}, \frac{g_j - g_{j-1}}{\Delta x}\right)$$

The numerical flux $\hat{F}_{j+\frac{1}{2}}$ is computed from an upwind flux splitting,

$$F = F^+ + F^-; \qquad F^{\pm}(U) = \frac{1}{2}(F(U) \pm aU), \quad a \geq \max|\text{eig } F'(U)|$$

and

$$\hat{F}_{j+\frac{1}{2}} = \hat{F}^+_{j+\frac{1}{2}} + \hat{F}^-_{j+\frac{1}{2}}$$

with a component-wise second order approximation,

$$\hat{F}^+_{j+\frac{1}{2}} = F_j^+ + s_j\Delta x/2, \qquad \hat{F}^-_{j+\frac{1}{2}} = F^-_{j+1} - s_{j+1}\Delta x/2$$

(a) Riemann shock tub perform with Sod initial data

$$U_L = \begin{pmatrix} 1 \\ 0 \\ 2.5 \end{pmatrix} \text{ if } 0 \le x < 0.5, \quad U_R = \begin{pmatrix} 0.125 \\ 0 \\ 0.25 \end{pmatrix} \text{ if } 0.5 \le x \le 1.$$

take a computational grids,

$$x_j = j\Delta x, \quad j = -2, -1, \cdots, N+2$$

where $\Delta x = 1/N$ and the boundary condition are taken as

$$\rho_{-2} = \rho_2, \quad \rho_{-1} = \rho_1; \quad m_{-2} = m_2, \quad m_{-1} = m_1; \quad E_{-2} = E_2, \quad E_{-1} = E_1;$$

Similar BC for $x = 1$.

Use 100, 200 and 400 points to compute the solution at $t = 0.1644$ and plot them together.

(b) Two interacting Blast Wave (Woodward-Colella). The initial data are

$$\begin{cases} \rho = 1, \ u = 0 \ \text{ for } 0 \le x \le 1, \\ p = 1000 \ \text{ for } 0 \le x \le 0.1; \ 0.01 \ \text{ for } 0.1 < x \le 0.9; \ 100 \ \text{ for } 0.9 < x \le 1. \end{cases}$$

and the boundary condition are taken as

$$\rho_{-2} = \rho_2, \quad \rho_{-1} = \rho_1; \quad m_{-2} = -m_2, \quad m_{-1} = -m_1; \quad E_{-2} = E_2, \quad E_{-1} = E_1;$$

Similar BC for $x = 1$.

Use 400 grid points to compute the numerical solution at $t = 0.03$ and $t = 0.038$ with $\Delta t = 0.00002$.

Use 800 grid points to compute the solution and compare your result at the resolution of 400 points.