Homework #2, due March 21

(1) Consider the following two-point boundary value problem,

$$-u'' = f(x), \quad \text{for } 0 \le x \le 1, \qquad u(0) = u(1) = 0 \tag{1}$$

and the following finite difference approximation,

$$-\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} = f(x_j), \qquad u(x_0) = u(x_N) = 0 \tag{2}$$

where grids x_i are give by

$$x_i = ih, \quad i = 0, 1, \cdots, N, \quad h = 1/N.$$

(a) For $\ell = 1, 2, \cdots N-1$, show that vector $u_\ell = \{\sin(j\ell h)\}_{j=1}^{N-1}$ is an eigenvector of the tridiagonal matrix in (2) and compute the corresponding eigenvalue.

(b) Write down weighted-Jacobi iteration with weight ω for (2) and reformulate it as a forward Euler time stepping for the heat equation

$$u_t = u_{xx} + f$$

with time step $\Delta t = \omega h^2/2$.

(c) For each eigenvector ℓ in **(a)**, estimate the convergence rate for the weighted Jacobi method. Find ω so that the convergence rate for $\ell = N/2, N/2+1, \cdots, N-1$ achieve the smallest rate.

(d) Write a multi-grid code for (2) and state in detail the steps in your algorithm. Plot the convergence history and estimate the convergence rate.

(2) Explore 1-d finite element methods

Consider the following two-point boundary value problem:

$$-(k(x)u')' + q(x)u = f(x), \quad u(0) = \alpha, \quad u(\pi) = \beta \tag{3}$$

where $k(x), q(x) > 0$. The grid discretization is given by

$$x_i = ih, \quad i = 0, 1, \cdots, N, \quad h = \pi/N. \tag{4}$$

The solution is written in terms of the basis function ϕ_i s.t.

$$u_h(x) = \sum_i u_i \phi_i$$

The governing equation can now be transformed into a system of linear equations by taking the inner product with the basis function ϕ_j, i.e,

$$\langle(-(ku_h')' + qu_h), \phi_j\rangle = \langle f, \phi_j\rangle$$

This can be written as a system of equations of the form

$$\sum_i a(\phi_i, \phi_j) u_i = \langle f, \phi_j\rangle, \quad j = 1, 2, \cdots, N-1$$

which can be solved for the unknowns u_i for a given set of basis functions ϕ_j. The coefficients of the matrix A are given by

$$a(\phi_i, \phi_j) = \int_0^\pi (k\phi_i'\phi_j' + a\phi_i\phi_j)\, dx$$

(a) Derive a linear finite element method approximation to the above differential equation and compare with the finite difference approximations in the first homework assignment,

$$-\frac{k(x_{j+1/2})(u(x_{j+1})-u(x_j))-k(x_{j-1/2})(u(x_j)-u(x_{j-1}))}{h^2}+q(x_j)u(x_j)=f(x_j), \quad (5)$$

(b) What is the general from of the above finite element approximation in a non-uniform grids? Write down the finite element approximation as a symmetric tridiagonal linear system with a coefficient matrix as $A = \text{diag }(a_i, b_i, b_i)$ and show that the tridiagonal matrix is diagonal dominant and positive definite.

(c) Write a program to perform Gaussian Elimination (or call LAPACK) and check accuracy as in the first homework assignment.

(d) Derive a quadratic finite element approximation of (3) and write a code to find the solution by Gaussian Elimination (or call LAPACK).

(e) Use the most suitable and efficient LAPACK routines to find the smallest and largest eigenvalues and find the conditions for the linear system.

(3) Explore the abstract form of finite element method

(a) Rewrite above finite element approximation in an abstract form: Find $u_h \in X_h$ such that $a(u_h, v_h) = f(v_h)$, for all $v_h \in X_h$. Expand the solution on the basis functions $u_h(x) = \sum_i u_i\phi_i(x)$ and find the linear systems.

(b) Show that u_h is indeed a projection of exact solution u in X_h space, i.e.,

$$a(u-u_h, v_h) = 0, \qquad \text{for all } v_h \in X_h \quad (6)$$

and show that u_h is also the minimizer of the

$$\Phi(u_h) = \min_{v_h \in X_h} \Phi(v_h), \quad \Phi(v_h) = \frac{1}{2}a(v_h, v_h) - f(v_h) \quad (7)$$

(4) Explore 2D finite element method

Consider the linear finite element approximation of Poisson equation

$$-\nabla^2 u(x,y) = f(x,y), \quad 0 < x, y < 2 \quad (8)$$

with the following grids:

(a) On the reference triangle with three vortices: (0,0), (0,1), (1,0), find the three basis functions which are one at one vortex and zero at the others. Compute the graduate of the basis functions and use it to compute the local stiffness matrix for

$$a_{i,j} = \int\int \nabla\phi_i \cdot \nabla\phi_j \, dxdy.$$

(b) To assemble the elements, consider the node (x_i, y_j) in the grid. The total stiffness matrix will contain contributions from all the surrounding elements.

(c) Derive a linear finite element method approximation and to show that it is same as the 5-point finite difference approximation.

(d) For $f(x,y) = (2 - \pi^2(x-2)^2)\sin\pi y$ and the boundary conditions:

$$u(x,0) = u(x,2) = u(2,y) = 0, \quad u(0,y) = -4\sin\pi y, \tag{9}$$

write a conjugate-gradient code to find the solution for a size of $h = 1/8, 1/16$ and $1/32$. Compare and check the accuracy for your result with the exact solution $u(x,y) = -(x-2)^2 \sin\pi y$. In your code, use a subroutine to compute the product of A with the vector, since this matrix is usually not given to you directly.

(5) Explore preconditional conjugate-gradient method

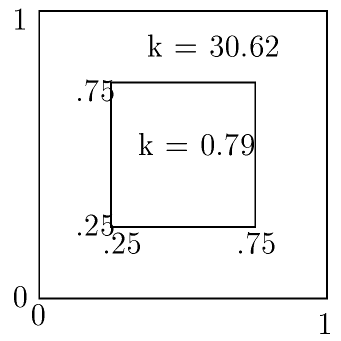
Use the finite difference or finite element method to approximate the following boundary value problem:

$$-\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) = 0, \quad \text{in } (0,1)\times(0,1), \tag{10}$$

with boundary condition

$$T(x,0) = 0, \quad T(x,1) = 1, \quad T(0,y) = T(1,y) = y, \tag{11}$$

The thermal coefficient is given by $k(x,y) = 0.79$ for $(x,y) \in (.25,.75)\times(.25,.75)$, and $k(x,y) = 30.62$ otherwise, as pictured below:



(a) Solve the system of linear equations using the conjugate gradient method, first with no preconditioner and then using the diagonal of the coefficient matrix as a preconditioner. Plot residual norm (divided by the norm of the initial residual) vs iteration number for each method. (Use a log scale for the residual norm; the MATLAB command *semilogy* will produce a such a plot.)

Try a few different mesh sizes, say, $h = 1/32$, $h = 1/64$, and $h = 1/128$. How is the number of iterations required to achieve a certain level of accuracy related to the mesh size?

(b) When $k(x,y)$ is a constant, say 1, the equation (10) becomes the standard Poisson equation. The 5-point finite difference approximation can be solved by fast Poisson solver. Use this fast Poisson solver as preconditioner to replace the diagonal preconditioner in **(a)** and compare the performance between these two preconditioners.