1. $\nabla f \cdot \frac{1}{\sqrt{3}}(1,-1,1)\Big|_{(1,0,-1)}$ in greatest

$\Rightarrow \nabla f = (t,-t,t)$ and $(t,-t,t)\cdot\frac{1}{\sqrt{3}}(1,-1,1) = 2 \Rightarrow \nabla f = \frac{2}{\sqrt{3}}(1,-1,1)$

(a) $\frac{2}{\sqrt{3}}(1,-1,1)\cdot\frac{1}{\sqrt{3}}(1,1,-1) = -\frac{2}{3}$ #

(b) $\frac{2}{\sqrt{3}}(x-1)+(-\frac{2}{\sqrt{3}})y+\frac{2}{\sqrt{3}}(z)=0 \Rightarrow x-y+z=1$ #

2. (a) Assume f and all its first and second derivatives are contd. in an open region R.

If, $f_x\Big|_{(a,b)} = f_y\Big|_{(a,b)} = 0$ and (i) $f_{xx}(a,b)>0$, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{vmatrix}_{(a,b)} > 0 \Rightarrow (a,b)$ is a min. pt.

(ii) $f_{xx}(a,b)<0$, $\begin{vmatrix}\end{vmatrix}_{(a,b)} > 0 \Rightarrow (a,b)$ " max pt.

(iii) $\begin{vmatrix}\end{vmatrix}_{(a,b)} < 0 \Rightarrow$ saddle.

(iv) $\begin{vmatrix}\end{vmatrix}_{(a,b)} = 0 \Rightarrow$ no conclusion.

(b) $(x-1)^2-(y-1)^2$.

3. ~~$f(x,y,z) = x^2+xy+2y^2-1$~~ ~~$\nabla f = (2x+y, 4y+x)$~~

(a) ~~$g(x,y,z) = x^2+y^2$~~ ~~$\nabla g = (2x, 2y)$~~

~~$\nabla g = \lambda \nabla f \Rightarrow \begin{cases} 2x = 2\lambda x + \lambda y \\ 2y = 4\lambda y + \lambda x \end{cases}$~~

3. (a) $f = x^2+xy+2y^2-1$

$g(x,y) = x^2+y^2$ $\Rightarrow$ $\begin{cases} \nabla f = (2x+y, 4y+x) \\ \nabla g = (2x, 2y) \end{cases}$

$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 2x+y = 2\lambda x \\ 4y+x = 2\lambda y \end{cases} \Rightarrow 2\lambda xy = 2x^2+y^2 = 4xy+x^2$

$\Rightarrow x^2+2xy = y^2$

$\Rightarrow 2+\frac{y}{x} = 4+\frac{x}{y}$

$\Rightarrow \frac{y}{x} = 1\pm\sqrt{2}$

$\lambda = 1+\frac{y}{2x} = \frac{3\pm\sqrt{2}}{2}$

$\Rightarrow \lambda = \frac{3\pm\sqrt{2}}{2} \Rightarrow y = (1\pm\sqrt{2})x$.

$x^2+x(y)\pm 2y^2 = 1 \Rightarrow 3y^2 = 1 \Rightarrow y = \pm\frac{\sqrt{3}}{3}$, $\Rightarrow x =$

$y = \frac{\sqrt{3}}{3}$, $x = \frac{-\sqrt{3}\pm\sqrt{6}}{3}$

$y = -\frac{\sqrt{3}}{3}$, $x = \frac{\sqrt{3}\pm\sqrt{6}}{3}$.

~~$f(x,(1+\sqrt{2})x) =$~~ $x^2+x((1+\sqrt{2})x)+2((1+\sqrt{2})x)^2 = 1 \Rightarrow x = \pm\sqrt{\frac{1}{8+5\sqrt{2}}} \Rightarrow y = \pm(1+\sqrt{2})\sqrt{\frac{1}{8+5\sqrt{2}}}$.

$x^2+x(1-\sqrt{2})x+2(1-\sqrt{2})^2x^2 = 1 \Rightarrow x = \pm\sqrt{\frac{1}{8-5\sqrt{2}}} \Rightarrow y = \pm(1-\sqrt{2})\frac{1}{\sqrt{8-5\sqrt{2}}}$

nearest: $\begin{cases} x = \pm\sqrt{\frac{1}{8+5\sqrt{2}}} \quad (y=(1+\sqrt{2})x \text{ or } x=(\sqrt{2}-1)y) \\ y = \pm(1+\sqrt{2})\frac{1}{\sqrt{8+5\sqrt{2}}} \end{cases}$ #

3. (b)

$$\begin{cases} f = xy + z^2 \\ g = y - 2x \\ h = x^2 + y^2 + z^2 - 1 \end{cases} \Rightarrow \begin{cases} \nabla f = (y, x, 2z) \\ \nabla g = (-2, 1, 0) \\ \nabla h = (2x, 2y, 2z) \end{cases}$$

$\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h \Rightarrow \begin{cases} y = -2\lambda_1 + 2\lambda_2 x \\ x = \lambda_1 + 2\lambda_2 y \\ 2z = 2\lambda_2 z \end{cases}$ (2) $\Rightarrow \begin{cases} \lambda_2 = 1 \\ y = 0 \\ x = \lambda_1 \end{cases}$ $\Rightarrow y - 2x = 0 \Rightarrow x = 0 \Leftarrow x^2 + y^2 = 1$

$f(0,0,\pm 1) = 1$ ← Max

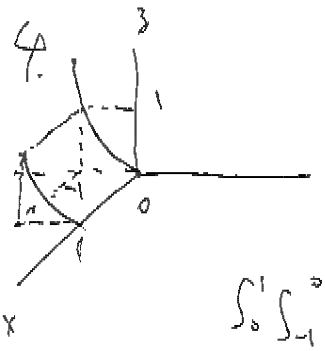
(ii) $z = 0$

$\begin{cases} y = -2\lambda_1 + 2\lambda_2 x \\ x = \lambda_1 + 2\lambda_2 y \end{cases}$ $\Rightarrow x = \pm\frac{\sqrt{5}}{5}, y = \mp\frac{2\sqrt{5}}{5}$, $\lambda_2 = x$

$\Rightarrow f(x,y,z) = -\frac{2}{5}$ at $\left(\frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}, 0\right)$ ← min

$\left(-\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, 0\right)$

4.



5 $x = \rho^{\frac{1}{3}} \cos\theta$

$y = \rho^{\frac{1}{3}} \sin\theta$

$$J = \frac{\partial(x,y)}{\partial(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{1}{3}\rho^{-\frac{2}{3}}\cos\theta & -\rho^{\frac{1}{3}}\sin\theta \\ \frac{1}{3}\rho^{-\frac{2}{3}}\sin\theta & +\rho^{\frac{1}{3}}\cos\theta \end{vmatrix}$$

$$= \frac{1}{3}\rho^{-\frac{1}{3}}.$$

6 (a) Normal: $\oint_{\partial R} M\,dy - N\,dx = \iint_R \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dx\,dy$

Tangent: $\oint_{\partial R} M\,dx + N\,dy = \iint_R \left(-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right) dx\,dy$

(b) $x = \cos t$

$y = 2\sin t$

$F = \begin{pmatrix} \cos t \\ -4\sin t \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix}$ $\quad 0 \le t \le 2\pi$

Normal $= \int_0^{2\pi} \cos t\,(+2\cos t) + 4\sin t\,(-\sin t)\,dt$

$= \int_0^{2\pi} (-1 + 3\cos 2\theta)\,d\theta = -2\pi$

$\iint_R (1-2)\,dA = -ab\pi = -2\pi$

Tangential: $\int_0^{2\pi} \cos t\,(-\sin t) + (-4\sin t)(2\cos t)\,dt = -\frac{9}{2}\int_0^{2\pi} \sin 2t\,dt = 0$

$\iint_R -\frac{\partial x}{\partial y} + \frac{\partial(-2y)}{\partial x}\,dA = \iint_R 0 = 0$

7. $\vec{F} = (\partial_x f, \partial_y f, \partial_z f)$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} = 0$$

8.

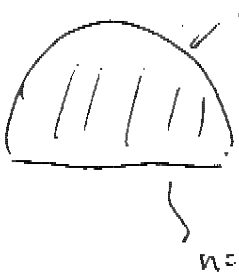
Direct computation: $d\sigma = \dfrac{dx\,dy}{(x,y,z)\cdot(0,0,1)} = \dfrac{dx\,dy}{\partial z}$

$$\iint_{x^2+y^2+z^2=1,\ z>0} z \; d\sigma$$

$$= \iint_{x^2+y^2=1} \frac{z}{z}\, d\sigma = \pi$$

div Thm: $z = (x,y,z)\cdot(0,0,1) = \vec{n}\cdot(0,0,1)$

$$\therefore \iint_{\substack{x^2+y^2+z^2=1 \\ z\geq 0}} z\, d\sigma + \iint_{x^2+y^2\leq 1} (0,0,1)\cdot(0,0,-1)\, dx\,dy$$



$$= \iiint \partial_z \operatorname{div}(0,0,1) = 0$$

$$\therefore \iint z\, d\sigma = \iint_{x^2+y^2\leq 1} 1\, dx\,dy = \pi$$