2D Case:

Let e be the error function that satisfies

$$\triangle_h e = \begin{cases} h, & \text{on the interface} \\ 0, & \text{elsewhere.} \end{cases}$$

Claim: $\|e\|_{L^2} = O(h^2 |\log h|)$.

Let $\triangle x = \triangle y = h = \frac{1}{N}$ and $x_i = ih, y_j = jh$. Define the O.N.B for the sine transformation

$$f_{k\ell}(x,y) = \frac{2}{N} \cdot \sin(kx\pi)\sin(\ell y\pi),$$

then $(f_{k\ell})_{ij} = f_{k\ell}(x_i, y_j) = \frac{2}{N}\sin(kx_i\pi)\sin(\ell y_j\pi)$.

Let G^h satisfies $\triangle_h G^h = \delta_{i_0,j_0}$, where $(i_0, j_0) \in$ interface. Then

$$< \triangle_h G^h, f_{k\ell} > = < \delta_{i_0,j_0}, f_{k\ell} > = (f_{k\ell})_{i_0,j_0}\,. \qquad (I)$$

On the other hand, we can write

$$\left(G^h\right)_{ij} = \sum_{k,\ell=1}^{N-1} \hat{G^h}_{k\ell}\,(f_{k\ell})_{ij}\,.$$

And hence

$$\begin{aligned} < \triangle_h G^h, f_{mn} > &= \sum_{i,j}\left(\sum_{k,\ell}(\lambda_k + \lambda_\ell)\,\hat{G^h}_{k\ell}\,(f_{k\ell})_{ij}\,(f_{mn})_{ij}\right) \\ &= (\lambda_m + \lambda_n)\,\hat{G^h}_{mn} \qquad (II) \end{aligned}$$

compare (I) and (II), we have

$$\hat{G^h}_{k\ell} = \frac{(f_{k\ell})_{i_0,j_0}}{(\lambda_k + \lambda_\ell)},$$

where $\lambda_k = \frac{-4\sin^2\left(\frac{kh\pi}{2}\right)}{h^2}$ is the eigenvalue for $\triangle_h$.

Thus

$$\begin{aligned} \left(G^h\right)_{ij} &= \sum_{k,\ell=1}^{N-1} \frac{(f_{k\ell})_{i_0,j_0}}{(\lambda_k + \lambda_\ell)} \cdot (f_{k\ell})_{ij} \\ &= \frac{-h^2}{N^2}\sum_{k,\ell=1}^{N-1} \frac{\sin(kx_i\pi)\sin(\ell y_j\pi)\cdot\sin(kx_{i_0}\pi)\sin(\ell y_{j_0}\pi)}{\sin^2\left(\frac{kh\pi}{2}\right) + \sin^2\left(\frac{\ell h\pi}{2}\right)} \end{aligned}$$

and then

$$
\begin{aligned}
\left|\left(G^h\right)_{ij}\right| &\leq \frac{h^2}{N^2}\sum_{k,\ell=1}^{N-1}\frac{1}{\sin\left(\frac{kh\pi}{2}\right)+\sin^2\left(\frac{\ell h\pi}{2}\right)} \\
&\sim \frac{1}{N^2}\int_h^1\int_h^1 \frac{dxdy}{\sin^2\left(\frac{\pi x}{2}\right)+\sin^2\left(\frac{\pi y}{2}\right)} \\
\left(\text{because of } \sin^2\left(\frac{\pi x}{2}\right)\geq x^2\right), \qquad &\leq \frac{1}{N^2}\int_0^{\frac{\pi}{2}}\int_{\sqrt{2}h}^{\sqrt{2}}\frac{rdrd\theta}{r^2}=\frac{\frac{\pi}{2}}{N_2}|\log h| \\
&= O\left(h^2|\log h|\right).
\end{aligned}
$$

Hence $\|e\|_{L^2}=O\left(hG^h\right)\cdot O(N)=O\left(G^h\right)=O\left(h^2|\log h|\right)$.

3D Case:

The mesh point is defined by $(x_i, y_j, z_k)=(ih, jh, kh)$, where $h=\frac{1}{N}$.

Let Γ be the interface in $\Omega\subset\mathbb{R}^3$, then

$$\#\{\text{ mesh points } \in\Gamma\}=O(N^2)$$

Also, let e be the error function with

$$\triangle_h e=\begin{cases} h, & \text{on} \quad \Gamma \\ 0, & \text{otherwise}\end{cases}$$

Claim: $\|e\|_{L^2(\Gamma)}=O(h)$

First, we define G^h satisfies

$$\triangle_h G^h=\delta_{i_0,j_0,k_0}, \qquad \text{for } (i_0,j_0,k_0)\in\Gamma.$$

Second, we define the O.N.B for the sine transformation

$$f_{\ell mn}=\left(\frac{2}{N}\right)^{\frac{3}{2}}\sin\left(\ell x\pi\right)\sin\left(my\pi\right)\sin\left(nz\pi\right),$$

then

$$(f_{\ell mn})_{ijk} = \left(\frac{2}{N}\right)^{\frac{3}{2}} \sin(\ell x_i \pi) \sin(m y_j \pi) \sin(n z_k \pi).$$

Note that

$$< \triangle_h G^h, f_{\ell mn} >= f_{\ell mn} \left(i_0, j_0, k_0\right), \tag{III}$$

On the other hand, we write

$$G^h(i,j,k) = \sum_{\ell mn} \hat{G}^h(\ell, m, n) f_{\ell mn}(i,j,k).$$

Therefore

$$\triangle_h G^h(i,j,k) = \sum_{\ell mn} \left(\lambda_\ell + \ell_m + \ell_n\right) \hat{G}^h \left(\ell, m, n\right) f_{\ell mn}(i,j,k)$$

and

$$< \triangle_h G^h, f_{\ell mn} >= \left(\lambda_\ell + \lambda_m + \lambda_n\right) \hat{G}^h \left(\ell, m, n\right). \tag{IV}$$

According to (III) and (IV), we have

$$\hat{G}^h(\ell, m, n) = \frac{f_{\ell mn}\left(i_0, j_0, k_0\right)}{\lambda_\ell + \lambda_m + \lambda_n}.$$

Therefore

$$\begin{aligned}
\left|G^h(i,j,k)\right| &\leq \sum_{\ell mn} \left| \frac{f_{\ell mn}(i_0, j_0, k_0)}{\lambda_\ell + \lambda_m + \lambda_n} f_{\ell mn}(i,j,k) \right| \\
&\leq \sum_{\ell mn} \frac{\left(\frac{2}{N}\right)^3 \cdot 1 \cdot h^2}{4\left[\sin^2\left(\frac{\ell h \pi}{2}\right) + \sin^2\left(\frac{m h \pi}{2}\right) + \sin^2\left(\frac{n h \pi}{2}\right)\right]} \\
&\simeq \frac{2}{hN^3} \int_h^1 \int_h^1 \int_h^1 \frac{dx dy dz}{x^2 + y^2 + z^2} \\
&\leq 2h^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_{\sqrt{3}h}^{\sqrt{3}} \frac{r^2 \sin\varphi}{r^2} dr d\varphi d\theta \\
&\approx O(h^2).
\end{aligned}$$

Finally $\|e\|_{L^2} = O(hG^h) \cdot O(N^2) = O(h)$.