L'Hôpital's Rule

Theorem 1 (L'Hôpital's Rule) *Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 6 $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

Theorem 2 (Strong Form of L'Hôpital's Rule) *Suppose that $f(a) = g(a) = 0$ and that f and g are differentiable on $(a-\delta, a+\delta)$. Suppose also that $g'(x) \neq 0$ if $x \neq a$. If*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \begin{cases} L \\ \infty \\ -\infty \end{cases},$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Hint: Theorem 2 can be derived easily (at least when the above limit is L), given the following

Theorem 3 *Cauchy's Mean Value Theorem Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$\begin{vmatrix} f(b) - f(a) & f'(c) \\ g(b) - g(a) & g'(c) \end{vmatrix} = 0.$$

Hint: Apply standard Mean Value Theorem to

$$F(x) = \begin{vmatrix} f(b) - f(a) & f(x) - f(a) \\ g(b) - g(a) & g(x) - g(a) \end{vmatrix} \qquad \text{on } [a, b].$$

Example 7 $\lim_{x\to 0} \frac{x-\sin x}{x^3} =$

Remark 1 *Under the same assumption above, if* $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *does not exist, it does NOT imply that*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \textit{ non-existent } .$$

Instead, L'Hôpital's Rule gives no conclusion in this case.

Example 8 $\lim_{x\to 0} \frac{x^2 \cos \frac{1}{x}}{\sin x} =$

Hint: L'Hôpital's Rule in inconclusive, use sandwich Theorem instead.

Variants of L'Hôpital's Rule (see the scanned reference for detail)

1. The one sided limit version.
2. The $\frac{\infty}{\infty}$ version.
3. The $\lim_{x\to\infty}$ versions.

In short, whenever you have a indefinite ratio of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, you can simply differentiate both the denominator and the enumerator until you get a limit (either finite or infinite).

Indefinite Differences and Products: $\infty - \infty$ and $0 \cdot \infty$

Example 9 *1.* $\lim_{x\to 0^+}(\frac{1}{\sin x} - \frac{1}{x}) =$

2. $\lim_{x\to\infty} x - \sqrt{x^2+x} =$

Remark 2 *The choice of writing $0 \cdot \infty$ as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ often makes a technical difference, as the following example shows:*

Example 10

$$\lim_{x\to 0^+} \frac{1}{x} \cdot e^{\frac{-1}{x}} = \lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x} = \lim_{x\to 0^+} \frac{x}{e^{\frac{1}{x}}}$$

Which one is better?

Intermediate Powers 1^∞, 0^0 and ∞^0

Example 11 *1.* $\lim_{x\to 0^+} x^{\frac{1}{x}} =$

2. $\lim_{x\to\infty} x^{\frac{1}{x}} =$

3. $\lim_{x\to 0^+} x^x =$

4. $\lim_{x\to 0^+}(1+ax)^{\frac{b}{x}} =$

Hint: always use the trick $x^y = (e^{\ln x})^y = (e^{y \ln x})$ and continuity of the exponential function: $e^{\lim f(x)} = \lim e^{f(x)}$.

Relative Rates of Growth, small o and Big O

Definition 1 (small o) *$f(x) = o(g(x))$ as $x \to a$ if $\lim_{x\to a} f(x)/g(x) = 0$. One can similarly define the case for $x \to \infty$. This means that $f(x)$ is genuinely "smaller" than $g(x)$.*

Definition 2 (Big O) *$f(x) = O(g(x))$ as $x \to a$ if $|\frac{f(x)}{g(x)}|$ is bounded (ie $\leq M$ for some $M > 0$) for all x sufficiently close to a (for all x large enough, in the $x \to \infty$ case). This means that $f(x)$ is "no larger" than $g(x)$.*

Example 12

1. $10x - 1 = O(x)$ *as* $x \to \infty$.
2. $ax^2 + bx + c = O(x^2)$ *as* $x \to \infty$.
3. $3x = O(\sqrt{x^2+1})$ *as $x \to$ anywhere. (Why?)*

Example 13

1. $\sin x = O(1)$ *as $x \to$ anywhere. (Why?)*
2. $\sin x = o(1)$ *as* $x \to 0$.
3. $\sin x = O(x)$ *as* $x \to 0$.

Example 14 *For any $a, b > 0$*

1. $\ln x = o(x^a)$ *and* $x = o(e^{bx})$ *as* $x \to \infty$.
2. $|\ln x| = o(x^{-a})$ *and* $x^{-1} = o(e^{\frac{b}{x}})$ *as* $x \to 0^+$.

Example 15 *Suppose $f(x)$ is differentiable at x_0 and let $L(x) = f(x_0) + f'(x_0)(x - x_0)$ be the linear approximation of f at x_0. Then*

1. $f(x) - L(x) = o(x - x_0)$ *as* $x \to x_0$.
2. *If in addition, f has continuous second derivative near x_0 (therefore f'' is bounded near x_0), then $f(x) - L(x) = O(|x - x_0|^2)$ as $x \to x_0$.*

Example 16 *If $f(x) = o(g(x))$, then $f(x) = O(g(x))$, but not vice versa.*

Example 17 *If $f(x) = O(|x - x_0|^2)$ as $x \to x_0$ then $f(x) = O(x - x_0)$ as $x \to x_0$, but not vice versa.*

Example 18 *If $f(x) = O(1)$ as $x \to \infty$ then $f(x) = O(x)$ as $x \to \infty$, but not vice versa.*

Inverse Trigonometric Functions and Their Derivatives

One of the main issue in defining inverse trigonometric functions is to restrict the domains of the original trigonometric functions. We truncate the domain for each of the trigonometric function so that the restricted function is one-to-one and maps to the same range as the un-restricted trigonometric functions.

Clearly, there are more than one way of truncating the domain to achieve the requirement.

We summarize below the conventional way of restriction for the trigonometric functions.

Proposition 7 *The following restricted trigonometric functions are one-to-one and maps onto the same range as the un-restricted ones*

1. $\sin : x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \longmapsto y \in [-1, 1]$.
2. $\cos : x \in [0, \pi] \longmapsto y \in [-1, 1]$.
3. $\tan : x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \longmapsto y \in \mathbb{R}$.
4. $\cot : x \in (0, \pi) \longmapsto y \in \mathbb{R}$.
5. $\sec : x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \longmapsto y \in (-\infty, -1] \cup [1, \infty)$.
6. $\csc : x \in [-\frac{\pi}{2}, 0) \cup (0, \pi] \longmapsto y \in (-\infty, -1] \cup [1, \infty)$.

As a corollary, the conventional domain and range for inverse trigonometric functions are given by

Corollary 1 *Domains and ranges of inverse trigonometric functions:*

1. $\sin^{-1} : y \in [-1, 1] \longmapsto x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
2. $\cos^{-1} : y \in [-1, 1] \longmapsto x \in [0, \pi]$.
3. $\tan^{-1} : y \in \mathbb{R} \longmapsto x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
4. $\cot^{-1} : y \in \mathbb{R} \longmapsto x \in (0, \pi)$.
5. $\sec^{-1} : y \in (-\infty, -1] \cup [1, \infty) \longmapsto x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
6. $\csc^{-1} : y \in (-\infty, -1] \cup [1, \infty) \longmapsto x \in [-\frac{\pi}{2}, 0) \cup (0, \pi]$.

For example, if $y \in [-1, 1]$ and $x = \sin^{-1} y$, then x is the unique element in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ that satisfies $\sin x = y$, etc.

Example 19 *For trigonometric and inverse trigonometric functions, the identity $f^{-1}(f(x)) = x$ may NOT hold for all $x \in \mathbb{R}$.*

1. $\cos(\cos^{-1}(\frac{2\pi}{3})) =?$
2. $\cos(\cos^{-1}(-\frac{2\pi}{3})) =?$

The derivatives of inverse trigonometric function is given by the general formula

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx}|_{x=f^{-1}(y)}}. \tag{1}$$

We will see that the domains of inverse trigonometric functions plays an essential role in the final step, namely expressing x in terms of y.

Example 20 *For $y \in (-1,1)$ (for derivative of $\sin^{-1}(y)$, we only need to consider y in the interior of the domain $[-1,1]$),*

$$\frac{d\sin^{-1} y}{dy} = \frac{1}{\frac{d\sin x}{dx}} = \frac{1}{\cos x} = \frac{1}{\pm\sqrt{1-y^2}} = \frac{1}{\sqrt{1-y^2}}.$$

Here we have used $x = \sin y$ in the third equality and selected the '+' sign in last equality since $\cos x > 0$ when $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which corresponds to $y \in (-1,1)$.

Similarly, we can also derive easily that

$$\frac{d\cos^{-1} y}{dy} = -\frac{1}{\sqrt{1-y^2}}, \qquad y \in (-1,1),$$

$$\frac{d\tan^{-1} y}{dy} = \frac{1}{1+y^2}, \qquad y \in \mathbb{R},$$

and

$$\frac{d\cot^{-1} y}{dy} = -\frac{1}{1+y^2}, \qquad y \in \mathbb{R}.$$

The next example is a little more complicated:

Example 21 *For $|y| > 1$, we have*

$$\frac{d\sec^{-1} y}{dy} = \frac{1}{\frac{d\sec x}{dx}} = \frac{1}{\sec x \tan x} = \frac{1}{\pm y\sqrt{y^2-1}} \tag{2}$$

where we have used

$$\tan x = \pm\sqrt{y^2-1}. \tag{3}$$

We now decide the sign in (3) and (2). The range of $\sec^{-1} y$, $|y| > 1$ is (consider interior points only) $x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Moreover,

$$y > 1 \iff x \in (0, \frac{\pi}{2}) \iff \tan x > 0 \iff \text{take '+' in (3)} \tag{4}$$

$$y < -1 \iff x \in (\frac{\pi}{2}, \pi) \iff \tan x < 0 \iff \text{take '-' in (3)} \tag{5}$$

From (4,5), it is easy to see that $\pm y = |y|$ in (2) and we conclude that

$$\frac{d\sec^{-1} y}{dy} = \frac{1}{|y|\sqrt{y^2-1}}$$

Similarly, we have

$$\frac{d\csc^{-1}y}{dy} = -\frac{1}{|y|\sqrt{y^2-1}}$$

The derivation is left as an exercise.

Using the chain rule, we can now compute the derivatives involving inverse trigonometric functions

Example 22 *1.* $\frac{d}{dx}\sin^{-1}(x^2) =$

2. $\frac{d}{dx}\tan^{-1}(\sin x) =$

Hyperbolic and Inverse Hyperbolic Functions and Their Derivatives

Definition 3 *Hyperbolic Functions*

1. $\sinh x = \frac{e^x - e^{-x}}{2}$
2. $\cosh x = \frac{e^x + e^{-x}}{2}$
3. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
6. $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

It is easy to derive the derivatives of the six hyperbolic functions, which are somewhat similar, but not the same as trigonometric functions:

Proposition 8 *Derivatives of hyperbolic functions:*

1. $\frac{d}{dx} \sinh x = \cosh x$
2. $\frac{d}{dx} \cosh x = \sinh x$
3. $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
4. $\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$
5. $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$
6. $\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$

By properly restricting the domains of hyperbolic functions, we can define the inverse hyperbolic functions:

Proposition 9 *Domains and ranges of inverse hyperbolic functions:*

1. $\sinh^{-1} : y \in \mathbb{R} \longmapsto x \in \mathbb{R}$.
2. $\cosh^{-1} : y \in [1, \infty) \longmapsto x \in [0, \infty)$.
3. $\tanh^{-1} : y \in (-1, 1) \longmapsto x \in \mathbb{R}$.
4. $\coth^{-1} : y \in (-\infty, -1) \cup (1, \infty) \longmapsto x \in (-\infty, 0) \cup (0, \infty)$.
5. $\operatorname{sech}^{-1} : y \in (0, 1] \longmapsto x \in [1, \infty)$.
6. $\operatorname{csch}^{-1} : y \in (-\infty, 0) \cup (0, \infty) \longmapsto x \in (-\infty, 0) \cup (0, \infty)$.

Using (1) and the following identities

Proposition 10 *1.* $\cosh^2 x - \sinh^2 x = 1$

2. $\tanh^2 x = 1 - \operatorname{sech}^2 x$

3. $\coth^2 x = 1 + \operatorname{csch}^2 x$

we can also derive the derivatives of inverse trigonometric functions

Proposition 11 *Derivatives of inverse hyperbolic functions:*

1. $\frac{d}{dx} \sinh^{-1} y = \frac{1}{\sqrt{1+y^2}}, \qquad y \in \mathbb{R}$

2. $\frac{d}{dx} \cosh^{-1} y = \frac{1}{\sqrt{y^2-1}}, \qquad y > 1$

3. $\frac{d}{dx} \tanh^{-1} y = \frac{1}{1-y^2}, \qquad y \in (-1, 1)$

4. $\frac{d}{dx} \coth^{-1} y = \frac{1}{1-y^2}, \qquad y \in (-\infty, -1) \cup (1, \infty)$

5. $\frac{d}{dx} \operatorname{sech}^{-1} y = -\frac{1}{y\sqrt{1-y^2}}, \qquad y \in (0, 1)$

6. $\frac{d}{dx} \operatorname{csch}^{-1} y = -\frac{1}{|y|\sqrt{1+y^2}}, \qquad y \in (-\infty, 0) \cup (0, \infty)$

The derivation of all the propositions in this section is left as an exercise.