Derivative of Transcendental Functions

Exponential Functions

For any rational number $a > 0$ and rational number x, the function

$$f(x) = a^x$$

can be defined.

Typical graphs of $y = a^x$ with $0 < a < 1$ and $a > 1$ are given like

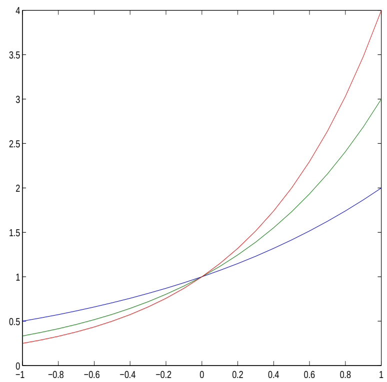

Figure 1: Plot of $y = a^x$, $a > 1$

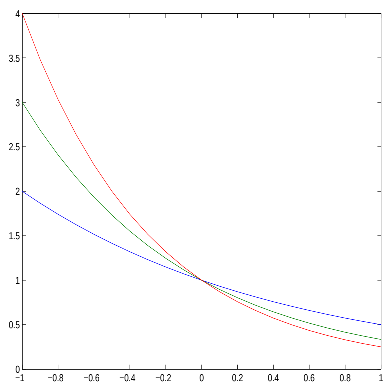

Figure 2: Plot of $y = a^x$, $0 < a < 1$

For a, $x \in \mathbb{R}$, $a > 0$, the values of a^x are defined by "continuous extension". The details are beyond this course.

Proposition 1 *For $a, b > 0$, $x, y \in \mathbb{R}$, we have*

- $a^x \cdot a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(\frac{1}{a})^x = a^{-x}$
- $(a^x)^y = a^{xy}$
- $a^x \cdot b^x = (ab)^x$
- $(\frac{a}{b})^x = \frac{a^x}{b^x}$

Proposition 2

$$\frac{d}{dx}a^x = \left(\lim_{h\to 0}\frac{a^h - 1}{h}\right) a^x$$

Suppose $\lim_{h\to 0}\frac{a^h-1}{h}$ exists, then the limit should be a functions of a, denoted by $g(a)$. It is easy to see that $g(1) = 0$ and that $g(a^2) = 2g(a)$, or more generally $g(a^b) = bg(a)$. Thus $g(a)$ is an increasing function of a with no upper or lower bounds.

We thus define the Euler number e to be one that satisfies

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

It is known that $e \sim 2.718281828\cdots$.

As a consequence, we have

$$\frac{d}{dx}e^x = e^x$$

Example 1 *Derivatives involving exponential functions*

1. $\frac{d}{dx}e^{kx} = ke^x$
2. $\frac{d}{dx}e^{x^2} = 2xe^x$
3. $\frac{d}{dx}e^{\sin x} = \cos xe^x$

How to compute the derivative of $y = a^x$?
The trick is to use the identity

$$a = e^{\log_e a}$$

we denote this special logarithmic function by $\ln = \log_e$

Proposition 3

$$\frac{d}{dx}a^x = \ln a \cdot a^x$$

Example 2 *More derivatives involving exponential functions*

1. $\frac{d}{dx}3^{-x} = -\ln 3 \cdot 3^x$
2. $\frac{d}{dx}3^{\sin x} =$
3. $\frac{d}{dx}(x^a + a^x) =$ *(Note: a power function (monomial) plus an exponential function)*
4. $\frac{d}{dx}a^{a^x} + a^{x^a} + x^{a^a} =$

Inverse Function of $y = f(x)$

A necessary and sufficient condition for

$$f : D_f \longrightarrow R_f \qquad (f \text{ maps from domain of } f \text{ to range of } f)$$

to have an inverse function is

"f is one-to-one and onto from domain of f to range of f"

If this is the case, we can define the inverse function

$$f^{-1} : R_f \longrightarrow D_f \qquad (f^{-1} \text{ maps from range of } f \text{ to domain of } f)$$

Proposition 4 *If the inverse functions of f exists, then*

- $f^{-1}(f(x)) = x$, *for all* $x \in D_f$.
- $f(f^{-1}(y) = y$, *for all* $y \in R_f$.

Notice that we have deliberately used a different notation (y) for the argument of f^{-1} to avoid possible confusion. This is different from the textbook.

It is better to use different letters (x and y) for elements in D_f and R_f. We will follow this notation through rest of this note.

The inverse function of $y = f(x)$ is thus denoted by $x = f^{-1}(y)$.

The exponential functions are one-to-one and onto from $\mathbb{R}$ to $\mathbb{R}^+$. The inverse function, denote by $\log_a$ maps from $\mathbb{R}^+$ to $\mathbb{R}$. Therefore

Proposition 5 *We have*

- $\log_a(a^x) = x$, *for all* $x \in \mathbb{R}$.
- $a^{\log_a y} = y$, *for all* $y \in \mathbb{R}^+$.

In particular,

- $\ln(e^x) = x$, *for all* $x \in \mathbb{R}$.
- $e^{\ln y} = y$, *for all* $y \in \mathbb{R}^+$.

Derivative of Inverse Functions and Logarithmic Functions

Since

$$f^{-1}(f(x)) = x \qquad \text{for all } x \in D_f,$$

we take the $x-$ derivative on both sides and use the chain rule to get

$$\frac{d}{dy}f^{-1}(f(x)) \cdot \frac{df(x)}{dx} = \frac{d}{dx}x = 1$$

In other words,

$$\frac{d}{dy}f^{-1}(y)|_{y=f(x)} \cdot (\frac{df(x)}{dx}) = \frac{d}{dx}x = 1$$

or

$$\frac{d}{dy}f^{-1}(y)|_{y=f(x)} = \frac{1}{\frac{df(x)}{dx}}$$

For example, if $f(x) = e^x$, then $f^{-1}(y) = \ln y$ and we have

$$\frac{d}{dy}\ln y|_{y=e^x} = \frac{1}{\frac{d}{dx}e^x} = \frac{1}{e^x} = \frac{1}{y} \qquad y > 0.$$

Note that the arguments y (of f^-1) and x (of f) are evaluated on different points: one on $f(x)$ and the other on x.

The following is WRONG due to confusion from bad notation:

$$\frac{d}{dx}\ln x = \frac{1}{\frac{d}{dx}e^x} = \frac{1}{e^x} = e^{-x}$$

Example 3 *Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of $\frac{df^{-1}(x)}{dx}$ at $x = -1 = f(3)$.*

Hint: to avoid confusion, it is better to change the problem to "Find the value of $\frac{df^{-1}(y)}{dy}$ at $y = -1 = f(3)$".

Proposition 6 *If $u(x) > 0$ is differentiable, then*

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$$

Example 4 *Derivative involving logarithmic functions*

1. $\frac{d}{dx} \ln(x^2 + 1) =$
2. $\frac{d}{dx} x^x =$ *(Hint: $x = e^{\ln x}$)*

Example 5 *(Derivative of rational functions by means of logarithmic functions). Find*

$$\frac{d}{dx} \frac{(x^2+1)(x-1)^{1/3}}{x+1}, \qquad x > 1$$

Hint: Let $y = \frac{(x^2+1)(x-1)^{1/3}}{x+1}$. Use the fact that $\frac{d}{dx} \ln y(x) = \frac{y'(x)}{y(x)}$.