Derivative of Transcendental Functions

Exponential Functions

For any rational number $a > 0$ and rational number x, the function

$$f(x) = a^x$$

can be defined.

Typical graphs of $y = a^x$ with $0 < a < 1$ and $a > 1$ are given like

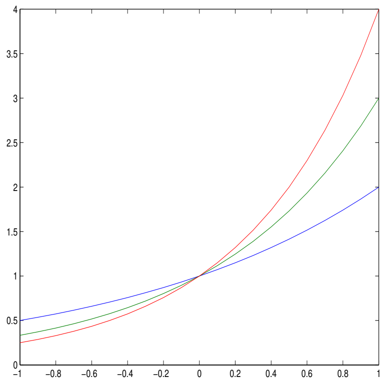

Figure 1: Plot of $y = a^x$, $a > 1$

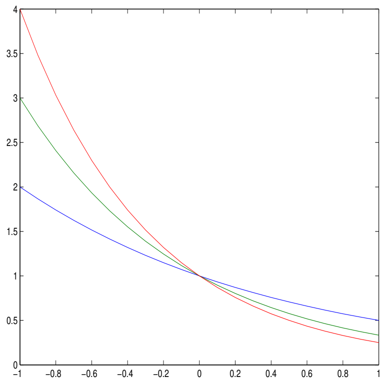

Figure 2: Plot of $y = a^x$, $0 < a < 1$

For a, $x \in \mathbb{R}$, $a > 0$, the values of a^x are defined by "continuous extension". The details are beyond this course.

Proposition 1 *For $a, b > 0$, $x, y \in \mathbb{R}$, we have*

- $a^x \cdot a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $(\frac{1}{a})^x = a^{-x}$
- $(a^x)^y = a^{xy}$
- $a^x \cdot b^x = (ab)^x$
- $(\frac{a}{b})^x = \frac{a^x}{b^x}$

Proposition 2

$$\frac{d}{dx}a^x = \left(\lim_{h\to 0}\frac{a^h - 1}{h}\right)a^x$$

Suppose $\lim_{h\to 0}\frac{a^h-1}{h}$ exists, then the limit should be a functions of a, denoted by $g(a)$. It is easy to see that $g(1) = 0$ and that $g(a^2) = 2g(a)$, or more generally $g(a^b) = bg(a)$. Thus $g(a)$ is an increasing function of a with no upper or lower bounds.

We thus define the Euler number e to be one that satisfies

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

It is known that $e \sim 2.718281828\cdots$.

As a consequence, we have

$$\frac{d}{dx}e^x = e^x$$

Example 1 *Derivatives involving exponential functions*

1. $\frac{d}{dx}e^{kx} = ke^x$
2. $\frac{d}{dx}e^{x^2} = 2xe^x$
3. $\frac{d}{dx}e^{\sin x} = e^x \cos x$

How to compute the derivative of $y = a^x$?
The trick is to use the identity

$$a = e^{\log_e a}$$

we denote this special logarithmic function by $\ln = \log_e$

Proposition 3

$$\frac{d}{dx} a^x = \ln a \cdot a^x$$

Example 2 *More derivatives involving exponential functions*

1. $\frac{d}{dx} 3^{-x} = -\ln 3 \cdot 3^x$
2. $\frac{d}{dx} 3^{\sin x} =$
3. $\frac{d}{dx} (x^a + a^x) =$ *(Note: a power function (monomial) plus an exponential function)*
4. $\frac{d}{dx} a^{a^x} + a^{x^a} + x^{a^a} =$

Inverse Function of $y = f(x)$

A necessary and sufficient condition for

$$f : D_f \longmapsto R_f \qquad (f \text{ maps from domain of } f \text{ to range of } f)$$

to have an inverse function is

"f is one-to-one and onto from domain of f to range of f"

If this is the case, we can define the inverse function

$$f^{-1} : R_f \longmapsto D_f \qquad (f^{-1} \text{ maps from range of } f \text{ to domain of } f)$$

Proposition 4 *If the inverse functions of f exists, then*

- $f^{-1}(f(x)) = x$, *for all* $x \in D_f$.
- $f(f^{-1}(y)) = y$, *for all* $y \in R_f$.

Notice that we have deliberately used a different notation (y) for the argument of f^{-1} to avoid possible confusion. This is different from the textbook.

It is better to use different letters (x and y) for elements in D_f and R_f. We will follow this notation through rest of this note.

The inverse function of $y = f(x)$ is thus denoted by $x = f^{-1}(y)$.

The exponential functions are one-to-one and onto from $\mathbb{R}$ to $\mathbb{R}^+$. The inverse function, denote by $\log_a$ maps from $\mathbb{R}^+$ to $\mathbb{R}$. Therefore

Proposition 5 *We have*

- $\log_a(a^x) = x$, *for all* $x \in \mathbb{R}$.
- $a^{\log_a y} = y$, *for all* $y \in \mathbb{R}^+$.

In particular,

- $\ln(e^x) = x$, *for all* $x \in \mathbb{R}$.
- $e^{\ln y} = y$, *for all* $y \in \mathbb{R}^+$.

Derivative of Inverse Functions and Logarithmic Functions

Since

$$f^{-1}(f(x)) = x \qquad \text{for all } x \in D_f,$$

we take the $x-$ derivative on both sides and use the chain rule to get

$$\frac{d}{dy} f^{-1}(f(x)) \cdot \frac{df(x)}{dx} = \frac{d}{dx} x = 1$$

In other words,

$$\frac{d}{dy} f^{-1}(y)|_{y=f(x)} \cdot (\frac{df(x)}{dx}) = \frac{d}{dx} x = 1$$

or

$$\frac{d}{dy} f^{-1}(y)|_{y=f(x)} = \frac{1}{\frac{df(x)}{dx}}$$

For example, if $f(x) = e^x$, then $f^{-1}(y) = \ln y$ and we have

$$\frac{d}{dy} \ln y|_{y=e^x} = \frac{1}{\frac{d}{dx} e^x} = \frac{1}{e^x} = \frac{1}{y} \qquad y > 0.$$

Note that the arguments y (of f^-1) and x (of f) are evaluated on different points: one on $f(x)$ and the other on x.

The following is WRONG due to confusion from bad notation:

$$\frac{d}{dx} \ln x = \frac{1}{\frac{d}{dx} e^x} = \frac{1}{e^x} = e^{-x}$$

Example 3 *Let $f(x) = x^3 - 3x^2 - 1, x \geq 2$. Find the value of $\frac{df^{-1}(x)}{dx}$ at $x = -1 = f(3)$.*

Hint: to avoid confusion, it is better to change the problem to "Find the value of $\frac{df^{-1}(y)}{dy}$ at $y = -1 = f(3)$".

Proposition 6 *If $u(x) > 0$ is differentiable, then*

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$$

Example 4 *Derivative involving logarithmic functions*

1. $\frac{d}{dx} \ln(x^2 + 1) =$
2. $\frac{d}{dx} x^x =$ *(Hint: $x = e^{\ln x}$)*

Example 5 *(Derivative of rational functions by means of logarithmic functions). Find*

$$\frac{d}{dx} \frac{(x^2+1)(x-1)^{1/3}}{x+1}, \qquad x > 1$$

Hint: Let $y = \frac{(x^2+1)(x-1)^{1/3}}{x+1}$. Use the fact that $\frac{d}{dx} \ln y(x) = \frac{y'(x)}{y(x)}$.

L'Hôpital's Rule

Theorem 1 (L'Hôpital's Rule) *Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Example 6 $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

Theorem 2 (Strong Form of L'Hôpital's Rule) *Suppose that $f(a) = g(a) = 0$ and that f and g are differentiable on $(a - \delta, a + \delta)$. Suppose also that $g'(x) \neq 0$ if $x \neq a$. If*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \left\{ \begin{array}{c} L \\ \infty \\ -\infty \end{array} \right. ,$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Hint: Theorem 2 can be derived easily (at least when the above limit is L), given the following

Theorem 3 *Cauchy's Mean Value Theorem Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that*

$$\left| \begin{array}{cc} f(b) - f(a) & f'(c) \\ g(b) - g(a) & g'(c) \end{array} \right| = 0.$$

Hint: Apply standard Mean Value Theorem to

$$F(x) = \left| \begin{array}{cc} f(b) - f(a) & f(x) - f(a) \\ g(b) - g(a) & g(x) - g(a) \end{array} \right| \qquad \text{on } [a, b].$$

Example 7 $\lim_{x\to 0} \frac{x-\sin x}{x^3} =$

Remark 1 *Under the same assumption above, if* $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *does not exist, it does NOT imply that*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \textit{ non-existent } .$$

Instead, L'Hôpital's Rule gives no conclusion in this case.

Example 8 $\lim_{x\to 0} \frac{x^2 \cos \frac{1}{x}}{\sin x} =$

Hint: L'Hôpital's Rule in inconclusive, use sandwich Theorem instead.

Variants of L'Hôpital's Rule (see the scanned reference for detail)

1. The one sided limit version.
2. The $\frac{\infty}{\infty}$ version.
3. The $\lim_{x\to\infty}$ versions.

In short, whenever you have a indefinite ratio of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, you can simply differentiate both the denominator and the enumerator until you get a limit (either finite or infinite).

Indefinite Differences and Products: $\infty - \infty$ and $0 \cdot \infty$

Example 9 *1.* $\lim_{x\to 0^+}(\frac{1}{\sin x} - \frac{1}{x}) =$

2. $\lim_{x\to\infty} x - \sqrt{x^2 + x} =$

Remark 2 *The choice of writing $0 \cdot \infty$ as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ often makes a technical difference, as the following example shows:*

Example 10

$$\lim_{x\to 0^+} \frac{1}{x} \cdot e^{\frac{-1}{x}} = \lim_{x\to 0^+} \frac{e^{\frac{-1}{x}}}{x} = \lim_{x\to 0^+} \frac{x}{e^{\frac{1}{x}}}$$

Which one is better?

Intermediate Powers 1^∞, 0^0 and ∞^0

Example 11 *1.* $\lim_{x\to 0^+} x^{\frac{1}{x}} =$

2. $\lim_{x\to\infty} x^{\frac{1}{x}} =$

3. $\lim_{x\to 0^+} x^x =$

4. $\lim_{x\to 0^+}(1 + ax)^{\frac{b}{x}} =$

Hint: always use the trick $x^y = (e^{\ln x})^y = (e^{y \ln x})$ and continuity of the exponential function: $e^{\lim f(x)} = \lim e^{f(x)}$.

Relative Rates of Growth, small o and Big O

Definition 1 (small o) *$f(x) = o(g(x))$ as $x \to a$ if $\lim_{x\to a} f(x)/g(x) = 0$. One can similarly define the case for $x \to \infty$. This means that $f(x)$ is genuinely "smaller" than $g(x)$.*

Definition 2 (Big O) *$f(x) = O(g(x))$ as $x \to a$ if $|\frac{f(x)}{g(x)}|$ is bounded (ie $\leq M$ for some $M > 0$) for all x sufficiently close to a (for all x large enough, in the $x \to \infty$ case). This means that $f(x)$ is "no larger" than $g(x)$.*

Example 12

1. *$10x - 1 = O(x)$ as $x \to \infty$.*
2. *$ax^2 + bx + c = O(x^2)$ as $x \to \infty$.*
3. *$3x = O(\sqrt{x^2+1})$ as $x \to$ anywhere. (Why?)*

Example 13

1. *$\sin x = O(1)$ as $x \to$ anywhere. (Why?)*
2. *$\sin x = o(1)$ as $x \to 0$.*
3. *$\sin x = O(x)$ as $x \to 0$.*

Example 14 *For any $a, b > 0$*

1. *$\ln x = o(x^a)$ and $x = o(e^{bx})$ as $x \to \infty$.*
2. *$|\ln x| = o(x^{-a})$ and $x^{-1} = o(e^{\frac{b}{x}})$ as $x \to 0^+$.*

Example 15 *Suppose $f(x)$ is differentiable at x_0 and let $L(x) = f(x_0) + f'(x_0)(x - x_0)$ be the linear approximation of f at x_0. Then*

1. *$f(x) - L(x) = o(x - x_0)$ as $x \to x_0$.*
2. *If in addition, f has continuous second derivative near x_0 (therefore f'' is bounded near x_0), then $f(x) - L(x) = O(|x - x_0|^2)$ as $x \to x_0$.*

Example 16 *If $f(x) = o(g(x))$, then $f(x) = O(g(x))$, but not vice versa.*

Example 17 *If $f(x) = O(|x - x_0|^2)$ as $x \to x_0$ then $f(x) = O(x - x_0)$ as $x \to x_0$, but not vice versa.*

Example 18 *If $f(x) = O(1)$ as $x \to \infty$ then $f(x) = O(x)$ as $x \to \infty$, but not vice versa.*

Inverse Trigonometric Functions and Their Derivatives

One of the main issue in defining inverse trigonometric functions is to restrict the domains of the original trigonometric functions. We truncate the domain for each of the trigonometric function so that the restricted function is one-to-one and maps to the same range as the un-restricted trigonometric functions.

Clearly, there are more than one way of truncating the domain to achieve the requirement.

We summarize below the conventional way of restriction for the trigonometric functions.

Proposition 7 *The following restricted trigonometric functions are one-to-one and maps onto the same range as the un-restricted ones*

1. $\sin : x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \longmapsto y \in [-1, 1]$.
2. $\cos : x \in [0, \pi] \longmapsto y \in [-1, 1]$.
3. $\tan : x \in (-\frac{\pi}{2}, \frac{\pi}{2}) \longmapsto y \in \mathbb{R}$.
4. $\cot : x \in (0, \pi) \longmapsto y \in \mathbb{R}$.
5. $\sec : x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi] \longmapsto y \in (-\infty, -1] \cup [1, \infty)$.
6. $\csc : x \in [-\frac{\pi}{2}, 0) \cup (0, \pi] \longmapsto y \in (-\infty, -1] \cup [1, \infty)$.

As a corollary, the conventional domain and range for inverse trigonometric functions are given by

Corollary 1 *Domains and ranges of inverse trigonometric functions:*

1. $\sin^{-1} : y \in [-1, 1] \longmapsto x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
2. $\cos^{-1} : y \in [-1, 1] \longmapsto x \in [0, \pi]$.
3. $\tan^{-1} : y \in \mathbb{R} \longmapsto x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
4. $\cot^{-1} : y \in \mathbb{R} \longmapsto x \in (0, \pi)$.
5. $\sec^{-1} : y \in (-\infty, -1] \cup [1, \infty) \longmapsto x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.
6. $\csc^{-1} : y \in (-\infty, -1] \cup [1, \infty) \longmapsto x \in [-\frac{\pi}{2}, 0) \cup (0, \pi]$.

For example, if $y \in [-1, 1]$ and $x = \sin^{-1} y$, then x is the unique element in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ that satisfies $\sin x = y$, etc.

Example 19 *For trigonometric and inverse trigonometric functions, the identity $f^{-1}(f(x)) = x$ may NOT hold for all $x \in \mathbb{R}$.*

1. $\cos(\cos^{-1}(\frac{2\pi}{3})) =?$
2. $\cos(\cos^{-1}(-\frac{2\pi}{3})) =?$

The derivatives of inverse trigonometric function is given by the general formula

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\frac{df(x)}{dx}|_{x=f^{-1}(y)}}. \tag{1}$$

We will see that the domains of inverse trigonometric functions plays an essential role in the final step, namely expressing x in terms of y.

Example 20 *For $y \in (-1,1)$ (for derivative of $\sin^{-1}(y)$, we only need to consider y in the interior of the domain $[-1,1]$),*

$$\frac{d\sin^{-1} y}{dy} = \frac{1}{\frac{d\sin x}{dx}} = \frac{1}{\cos x} = \frac{1}{\pm\sqrt{1-y^2}} = \frac{1}{\sqrt{1-y^2}}.$$

Here we have used $x = \sin y$ in the third equality and selected the '+' sign in last equality since $\cos x > 0$ when $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which corresponds to $y \in (-1,1)$.

Similarly, we can also derive easily that

$$\frac{d\cos^{-1} y}{dy} = -\frac{1}{\sqrt{1-y^2}}, \qquad y \in (-1,1),$$

$$\frac{d\tan^{-1} y}{dy} = \frac{1}{1+y^2}, \qquad y \in \mathbb{R},$$

and

$$\frac{d\cot^{-1} y}{dy} = -\frac{1}{1+y^2}, \qquad y \in \mathbb{R}.$$

The next example is a little more complicated:

Example 21 *For $|y| > 1$, we have*

$$\frac{d\sec^{-1} y}{dy} = \frac{1}{\frac{d\sec x}{dx}} = \frac{1}{\sec x \tan x} = \frac{1}{\pm y\sqrt{y^2-1}} \tag{2}$$

where we have used

$$\tan x = \pm\sqrt{y^2-1}. \tag{3}$$

We now decide the sign in (3) and (2). The range of $\sec^{-1} y$, $|y| > 1$ is (consider interior points only) $x \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi)$. Moreover,

$$y > 1 \iff x \in (0, \frac{\pi}{2}) \iff \tan x > 0 \iff \text{take '+' in (3)} \tag{4}$$

$$y < -1 \iff x \in (\frac{\pi}{2}, \pi) \iff \tan x < 0 \iff \text{take '-' in (3)} \tag{5}$$

From (4,5), it is easy to see that $\pm y = |y|$ in (2) and we conclude that

$$\frac{d\sec^{-1} y}{dy} = \frac{1}{|y|\sqrt{y^2-1}}$$

Similarly, we have

$$\frac{d\csc^{-1} y}{dy} = -\frac{1}{|y|\sqrt{y^2-1}}$$

The derivation is left as an exercise.

Using the chain rule, we can now compute the derivatives involving inverse trigonometric functions

Example 22 *1.* $\frac{d}{dx}\sin^{-1}(x^2) =$

2. $\frac{d}{dx}\tan^{-1}(\sin x) =$

Hyperbolic and Inverse Hyperbolic Functions and Their Derivatives

Definition 3 *Hyperbolic Functions*

1. $\sinh x = \frac{e^x - e^{-x}}{2}$
2. $\cosh x = \frac{e^x + e^{-x}}{2}$
3. $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
4. $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$
5. $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$
6. $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

It is easy to derive the derivatives of the six hyperbolic functions, which are somewhat similar, but not the same as trigonometric functions:

Proposition 8 *Derivatives of hyperbolic functions:*

1. $\frac{d}{dx} \sinh x = \cosh x$
2. $\frac{d}{dx} \cosh x = \sinh x$
3. $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$
4. $\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$
5. $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x \tanh x$
6. $\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x \coth x$

By properly restricting the domains of hyperbolic functions, we can define the inverse hyperbolic functions:

Proposition 9 *Domains and ranges of inverse hyperbolic functions:*

1. $\sinh^{-1} : y \in \mathbb{R} \longmapsto x \in \mathbb{R}$.
2. $\cosh^{-1} : y \in [1, \infty) \longmapsto x \in [0, \infty)$.
3. $\tanh^{-1} : y \in (-1, 1) \longmapsto x \in \mathbb{R}$.
4. $\coth^{-1} : y \in (-\infty, -1) \cup (1, \infty) \longmapsto x \in (-\infty, 0) \cup (0, \infty)$.
5. $\operatorname{sech}^{-1} : y \in (0, 1] \longmapsto x \in [1, \infty)$.
6. $\operatorname{csch}^{-1} : y \in (-\infty, 0) \cup (0, \infty) \longmapsto x \in (-\infty, 0) \cup (0, \infty)$.

Using (1) and the following identities

Proposition 10 *1.* $\cosh^2 x - \sinh^2 x = 1$

2. $\tanh^2 x = 1 - \operatorname{sech}^2 x$

3. $\coth^2 x = 1 + \operatorname{csch}^2 x$

we can also derive the derivatives of inverse trigonometric functions

Proposition 11 *Derivatives of inverse hyperbolic functions:*

1. $\frac{d}{dx} \sinh^{-1} y = \frac{1}{\sqrt{1+y^2}}, \qquad y \in \mathbb{R}$

2. $\frac{d}{dx} \cosh^{-1} y = \frac{1}{\sqrt{y^2-1}}, \qquad y > 1$

3. $\frac{d}{dx} \tanh^{-1} y = \frac{1}{1-y^2}, \qquad y \in (-1, 1)$

4. $\frac{d}{dx} \coth^{-1} y = \frac{1}{1-y^2}, \qquad y \in (-\infty, -1) \cup (1, \infty)$

5. $\frac{d}{dx} \operatorname{sech}^{-1} y = -\frac{1}{y\sqrt{1-y^2}}, \qquad y \in (0, 1)$

6. $\frac{d}{dx} \operatorname{csch}^{-1} y = -\frac{1}{|y|\sqrt{1+y^2}}, \qquad y \in (-\infty, 0) \cup (0, \infty)$

The derivation of all the propositions in this section is left as an exercise.