

Integration

Integration by Substitution: the Chain Rule

$$\int f(g(x)) \cdot g'(x) dx = F(g(x)) + C,$$

where F is an antiderivative of f . That is, $\frac{dF(u)}{du} = f(u)$.

Typical Application:

1.

$$\int f(x^{n+1})x^n dx =$$

$$\int f(P(x))P'(x)dx =$$

2.

$$\int f(\cos(x)) \sin(x) dx =$$

$$\int f(\sin(x)) \cos(x) dx =$$

3.

$$\int f(\tan(x)) \sec^2(x) dx =$$

$$\int f(\cot(x)) \csc^2(x) dx =$$

4.

$$\int f(\sec(x)) \sec(x) \tan(x) dx =$$

$$\int f(\csc(x)) \csc(x) \cot(x) dx =$$

How to apply the chain rule? Take the first equation above for example:

Method 1: slow, step by step method:

Introduce $u = x^{n+1}$, $du = (n+1)x^n$.

Method 2: faster, using a convenient notation:

Write $x^n = \frac{1}{n+1}d(x^{n+1})$.

Example 1

$$\int (x + 2)^5 dx$$

Example 2

$$\int \sqrt{1 + y^2} y \, dy$$

Example 3

$$\int \sqrt{4t - 1} dt$$

Example 4

$$\int \cos(7\theta + 5) d\theta$$

Example 5

$$\int x^2 \sin(x^3) dx$$

Example 6

$$\int \frac{1}{\cos^2 2\theta} d\theta$$

Example 7

$$\int (x^2 + 2x - 3)^2 (x + 1) dx$$

Example 8

$$\int \sin^4 t \cos t dt$$

Example 9

$$\int \frac{z dz}{\sqrt[3]{z^2 + 1}}$$

Substitution in Definite Integrals

$$\int_a^b f(g(x)) \cdot g'(x) dx = F(u) \Big|_{u=g(a)}^{u=g(b)}, \quad \text{or} \quad = F(g(x)) \Big|_{x=a}^{x=b}$$

Example 10

$$\int_{-1}^1 x^2 \sqrt{x^3 + 1} dx$$

Example 11

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot \theta \csc^2 \theta d\theta$$