Application of Derivatives

Basic Graphing

How to graph $y = f(x)$**?**

1. Find f' and f''.
2. Determine where the graph of $f(x)$ is increasing or decreasing from f'
3. Determine concavity of the graph from f''
4. Plot specific points (roots, critical points and inflection points) and sketch the curve.

Remark 1 *A reflection point is a point* x_0 *where* $f'(x_0) \in \mathbb{R} \cup \{\pm\infty\}$ *and* f *changes concavity at* x_0*.* $f''(x_0)$ *may or may not exist. For example,* $f(x) = x|x|$ *and* $x_0 = 0$*. Moreover, if* $f''(x_0) = 0$*, it may or may not be a point of reflection. For example,* $f(x) = x^4$ *and* $x_0 = 0$*.*

Example 1 $f(x) = x^{1/3}(x - 4)$

$$f'(x) = \frac{4}{3}x^{-2/3}(x-1)$$
$$f''(x) = \frac{4}{9}x^{-5/3}(x+2)$$

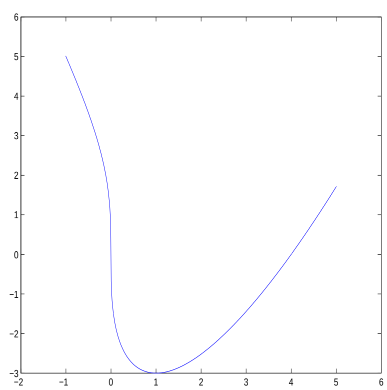

Figure 1: Plot of $y = x^{1/3}(x - 4)$

Example 2 $f(x) = x^4 - 4x^3 + 10$

$$f'(x) = 4x^2(x-3)$$
$$f''(x) = 12x(x-2)$$

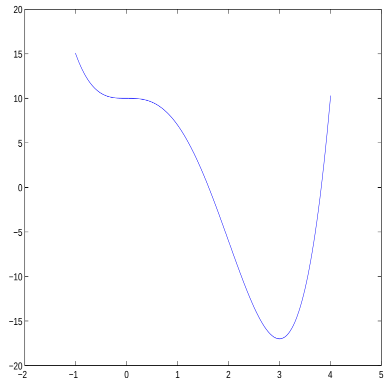

Figure 2: Plot of $y = x^4 - 4x^3 + 10$

Example 3 $f(x) = x^{5/3} - 5x^{2/3}$

$$f'(x) = \frac{5}{3}x^{-1/3}(x-2)$$
$$f''(x) = \frac{10}{9}x^{-4/3}(x+1)$$

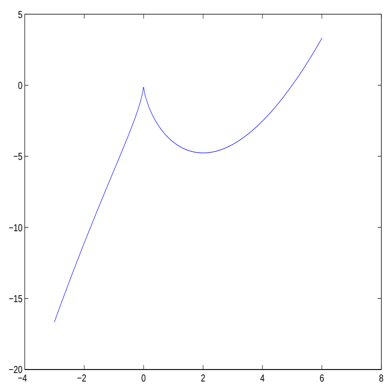

Figure 3: Plot of $y = x^{5/3} - 5x^{2/3}$

Asymptotes and Dominating Terms

Definition 1 *The line $y = b$ is called a Horizontal Asymptote of $y = f(x)$ if*

$$\lim_{x\to\infty} f(x) = b \qquad \textit{or} \qquad \lim_{x\to-\infty} f(x) = b$$

The line $x = a$ is called a Vertical Asymptote of $y = f(x)$ if

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \textit{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty$$

Example 4 $y = \frac{1}{x^2-1}$

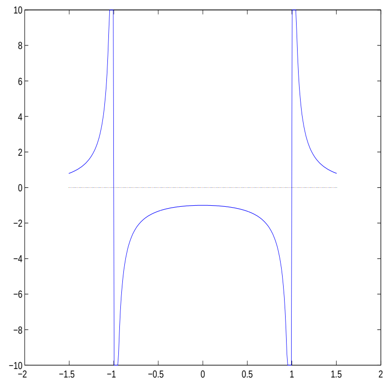

Figure 4: Plot of $y = \frac{1}{x^2-1}$

Definition 2 *The line $y = ax + b$ is called an Oblique Asymptote of $y = f(x)$ if*

$$\lim_{x \to \infty} f(x) - (ax + b) = 0 \qquad or \qquad \lim_{x \to -\infty} f(x) - (ax + b) = 0$$

Example 5 $y = \frac{x^2-3}{x-1}$

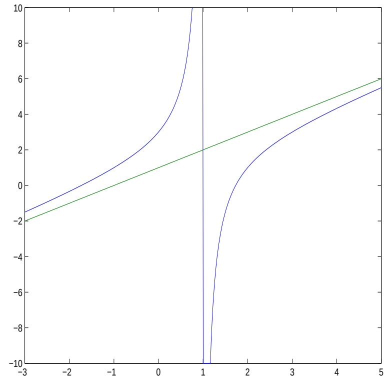

Figure 5: Plot of $y = \frac{x^2-3}{x-1}$

Similarly, we say that $g(x)$ is a Dominant Term of $f(x)$ near $x = c, \pm\infty$ if

$$\lim_{x \to c, \pm\infty} \frac{f(x) - g(x)}{g(x)} = 0$$

Example 6 $y = \frac{x^3+1}{x}$

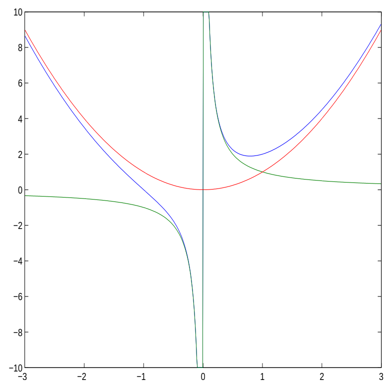

Figure 6: Plot of $y = \frac{x^3+1}{x}$

Optimization-Finding Absolute Extrema (Skip Examples in Economics)

How to Find Global Min and Global Max of $y = f(x)$?

1. Find all critical points of f.

2. Compare values of $f(x)$ at all critical points and endpoints.

Example 7 *Find the largest box (in volume) made from a 10-by-10-cm sheet with 4 small squares of equal size cut at the 4 corners.*

Example 8 *Find a rectangle contained in the ellipse $x^2 + 2y^2 = 1$ that has the largest area and perimeter, respectively.*

The Mean Value Theorem

Theorem 1 (From Advanced Calculus) *A continuous function f defined on a closed interval $[a,b]$ always attains (absolute) maximum and absolute minimum.*

It is easy to see that one may fail to find absolute maximum and minimum proveided either f is not continuous or the interval is not closed, for example (a,b).

Example 9 $y = 1/x$ *on* $[-1,1]$

Example 10 $y = 1/x$ *on* $(0,1]$

Theorem 2 (Rolle's Theorem) *If g is contoinoues on $[a.b]$ and differentiable on (a,b), and if $g(a) = g(b) = 0$, then there exists a point $c \in (a,b)$ such that $g'(c) = 0$*

Rolle's Theorem a special case of the following

Theorem 3 (Mean Value Theorem) *If f is contoinoues on $[a.b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.*

In fact, it is easy to show that Rolle's Theorem is equivalent to Mean Value Theorem. In fact, if we take $g(x) = f(x) - (f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$, it is easy to see that Mean Value Theorem can be derived from Rolle's Theorem.

Corollary 1 *If $f' = 0$ on an interval (a, b), then $f(x)$ is a constant on (a, b).*

Corollary 2 *If $f' = g'$ on an interval (a, b), then $f(x) - g(x)$ is a constant on (a, b).*

Corollary 3 *Suppose f is contoinoues on $[a.b]$ and differentiable on (a, b). If $f' > 0$ on (a, b), then f strictly increasing on $[a, b]$. If $f' < 0$ on (a, b), then f strictly decreasing on $[a, b]$.*

Remark 2 (see Chap 9, Taylor's Theorem) *Applying the Mean Value Theorem repeatedly, one can show that*

$$f(b) = f(a) + f'(a)(g - a) + \frac{1}{2}f''(y)(b - a)^2$$

for some y between a and b, provided f, f' are continuous on $[a, b]$ and f is twice differentiable on (a, b).

proof :
Define a quadratic function

$$g(x) = f(a) + f'(a)(x - a) + \frac{K}{2}(x - a)^2.$$

Note that $f(a) = g(a)$, $f'(a) = g'(a)$. We choose the constant K such that $f(b) = g(b)$. It follows that we can apply the Mean Value Theorem to $f(x) - g(x)$. (twice, on different intervals. details omitted) $\square$

Example 11 *Show that $f(x) = x^3 + 3x + 1$ has exactly one real root.*