Suggested Reading 2.1-2.3

Informal Definition of Limit (of function's value)

If $f(x) \longrightarrow L$ when $x \to c$

we say $\lim_{x \to c} f(x) = L$

Remark: The definition of $\lim_{x \to c} f(x)$ is not related to "$f(c)$".

Examples: (1) $f(x) = x$

(2) $f(x) = k$ (constant function)

(3) $f(x)$ = polynomials and rational functions

Properties of limits

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ both exists (and finite)

Then

- $\lim_{x \to c} \left(f(x) \pm g(x) \right) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
- $\lim_{x \to c} \left(f(x) \cdot g(x) \right) = \left(\lim_{x \to c} f(x) \right) \cdot \left(\lim_{x \to c} g(x) \right)$
- $\lim_{x \to c} \left(k f(x) \right) = k \lim_{x \to c} f(x)$
- $\lim_{x \to c} \left(\dfrac{f(x)}{g(x)} \right) = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ Provided $\lim_{x \to c} g(x) \neq 0$

Corollary: (1), (2) $\Rightarrow$ (3)

Examples of "$\lim_{x \to c} f(x)$ does not exist"

(1) $f(x) = \lfloor x \rfloor$ = integer part of x

i.e. $f(x) = n$ iff $n \leq x < n+1$

↑ (if and only if)

integer

(2) ~~$f(x)$~~

"$f(x) = \sin\left(\frac{1}{x}\right)$, $x \neq 0$ ($f(0)$ is not relevant)

In (1): we have $\lim_{x \to n^-} f(x) = n-1$

while $\lim_{x \to n^+} f(x) = n$

$\lim_{x \to n^-} \neq \lim_{x \to n^+}$ so $\lim_{x \to n}$ does not exist.

In (2) $f\left(\frac{1}{n\pi}\right) = 0$ $n = 1, 2, 3, \ldots$

while $f\left(\frac{1}{(2n \pm \frac{1}{2})\pi}\right) = \pm 1$,

values of $f(x)$ oscillates between $[-1, 1]$

as $x \to 0$.

Example (3):

$\lim_{x \to 0} \frac{1}{x^2} = ?$

We call it ∞ instead of non-existent.

But $\lim_{x \to 0} \frac{1}{x}$ does not exist; (Why?)

Limits involving Infinity

$x \to \infty$ means x gets larger and larger

$x \to -\infty$ means x gets smaller and smaller
(e taking account the sign)
(i.e. ~~large and negative~~)
(i.e. negative with large absolute value)

Examples
- $\lim_{x \to \infty} \frac{1}{x} = 0$ ("0^+", to be more specific)
- $\lim_{x \to 0^+} \frac{1}{x} = \infty$
- $\lim_{x \to 0^-} \frac{1}{x} = -\infty$
- $\lim_{x \to -\infty} \frac{1}{x} = 0^-$
- Properties of limits on Page 2-1 remains valid if $x \to c$ is replaced by $x \to \infty$ or $x \to -\infty$

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Limits of rational functions as $x \to \pm\infty$

$$\lim_{x \to \infty} \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0} = \frac{\pm\infty}{\pm\infty} = ?$$

Trick: Find $p = \max(m, n)$ and devide by x^p on numerator and denominator

Conclusion:

- $n < m$, $\lim_{x \to \pm\infty} \frac{(\ )}{(\ )} = 0$
- $n = m$, $\lim_{x \to \pm\infty} \frac{(\ )}{(\ )} = \frac{a_n}{b_n}$
- $n > m$, $\lim_{x \to \pm\infty} \frac{(\ )}{(\ )} = \infty$ or $-\infty$

Definition of "$\lim_{x\to c} f(x) = L$" Using ε, δ

How to specify "as $x \to c$" and "$f(x) \to L$" ?

and "$x \to c$" implies "$f(x) \to L$"

or "$f(x) \to L$ as $x \to c$" ?

Ans: Definition of $\lim_{x\to c} f(x) = L$

~~Ans~~: Given $\varepsilon > 0$, we can always find a $\delta > 0$ such that

"$0 < |x-c| < \delta$ implies $|f(x) - L| < \varepsilon$"

Remark: (1) This definition is "operational", to verify whether $\lim_{x\to c} f(x) = L$, one needs to develop a strategy to find a $\delta > 0$ for any given $\varepsilon > 0$

(2) We do not care whether $f(c) = L$, therefore we use "$0 < |x-c| < \delta$" instead of "$|x-c| < \delta$"

If we use "$|x-c| < \delta$", the definition will impose "$\lim_{x\to c} f(x) = L$" AND "$f(c) = L$"

Example: Verify $\lim_{x \to 6} x^2 = 6^2$

Step 1. given any $\varepsilon > 0$, find an interval around 6 on which $|x^2 - 6^2| < \varepsilon$, that is $\sqrt{6^2-\varepsilon} < x < \sqrt{6^2+\varepsilon}$

(we may assume ε small so that $6^2 - \varepsilon > 0$)

Step 2: Try to fit @ an interval of the form

$(6-\delta, 6) \cup (6, 6+\delta)$ { that is, $0 < |x-6| < \delta$ }

inside $(\sqrt{6^2-\varepsilon}, \sqrt{6^2+\varepsilon})$

For this purpose, it suffices to take

$\delta = \min\{6 - \sqrt{6^2-\varepsilon}, \sqrt{6^2+\varepsilon} - 6\}$

(or any smaller positive number)

Step 3 Check indeed that

$0 < |x-6| < \delta$ implies $|x^2 - 6^2| < \varepsilon$

Detail left as exercise.

The Sandwich Theorem.

Suppose $g(x) \le f(x) \le h(x)$

for $x \in (c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$

and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$

Then $\lim_{x \to c} f(x) = L$

Examples

(1) $\sin\theta < \theta < \tan\theta$ $\quad \theta > 0$

$\Rightarrow \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

(2) $\lim_{x \to 0} x \sin\frac{1}{x} < \lim_{x \to 0} \frac{\sin\frac{1}{x}}{\frac{1}{x}} \to 0$

Continuity

Continuity at a point

$f : [a,b] \longrightarrow \mathbb{R}$.

If $c \in (a,b)$, $f(x)$ is continuous at c iff

$$\lim_{x \to c} f(x) = f(c)$$

($f(x)$ is continuous at a iff

$$\lim_{x \to a^+} f(x) = f(a)$$

$f(x)$ is continuous at b iff

$$\lim_{x \to b^-} f(x) = f(b) \ )$$

Remark: $\lim_{x \to c} f(x) = f(c)$ means

(1) f is defined on c

(2) $\lim_{x \to c} f(x)$ exists

(3) they are equal.

~~Typical cases of failure of~~

Typical cases of Non-continuity

(1). f is not defined on c

$f(x) = \sin\frac{1}{x}$. $x \neq 0$

$f(0)$ not defined.

(2) $\lim_{x\to c} f(x)$ does not exist.

- $f(x) = \sin\frac{1}{x}$, $c = 0$
- $f(x) = [x]$, $x = n$.

(3) $\lim_{x\to c} f(x)$ exists, but $\neq f(c)$

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 1 & x = 0 \end{cases}$$

It is possible to Redefine $f(c)$ so in case (3) to make $f(x)$ continuous at c

$f(c)$ def ?

Prop: f and g are continues at c.

Then so is $f \pm g$, $f \cdot g$, $k \cdot g$ and f/g (If $g(c) \neq 0$)

Corollary: continuity of the constant function $f(x)=c$ and identity function $f(x)=x$ implies the continuity of polynomials and rational functions (if denominator $\neq 0$)

Prop: (Continuity of Composition of functions)

Suppose $f(x)$ is continuous at $x=c$

$g(y)$ is continuous at $y=f(c)$

Then $g\circ f$ is continuous at $x=c$

Pf: Left as an exercise for practice

Examples (1) $f(x)=\begin{cases}1, & x \text{ rational}\\ 0, & x \text{ irrational}\end{cases}$

$\Rightarrow f(x)$ is discontinuous everywhere

(2) $\sqrt{x\cos x}$ is continuous at $x=\frac{\pi}{3}$

from the two Propositions above.

Defining continuity in terms of ε and δ.

Given $\varepsilon > 0$, $\exists\ \delta > 0$ (δ may depend on ε) such that

"$|x-c| < \delta \implies |f(x) - f(c)| < \varepsilon$"

Remark: $|x-c| < \delta$ can be replaced by $0 < |x-c| < \delta$ here, the results ⓐ are the same.

Q: Why bother to define continuity using ε, δ?

Remark: A not so obvious fact (Beyond this course)

$f_1(x), f_2(x), \cdots f_n(x), \cdots$ defined on (a,b)

Suppose each $f_n(x)$ is continuous on (a,b)

and If $\lim_{N\to\infty} \sum_{i=1}^{N} f_i(x) = g(x)$ exists for each $x \in (a,b)$, then $g(x)$ may or may not be continuous!

Examples (Also beyond this course!)

(1) $\sum_{i=1}^{n} f_i(x) = x^n$ on $[0,1]$

(2) $\sum_{i=1}^{n} f_i(x) = \tan^{-1}(nx)$ on $\mathbb{R}$

Examples: Prove $f+g$ is continuous at c if both f and g are.

(Detail: see 2.5)

Q: Where does the argument fail if one tries to repeat the proof on

$$\sum_{n=1}^{\infty} f_1(x)+f_2(x)+\cdots+f_n(x)+\cdots = g(x)$$

mentioned on page 2-11?

Application of continuity

Intermediate value Theorem: If $f(x)$ is cont. on $[a,b]$ $\Rightarrow$ f takes any value between $f(a)$ and $f(b)$

(Rem: " " $\Rightarrow$ " ")

Homework: find an example (with sketching)

Corollary: f and g are continuous on $[a,b]$

$f(a) < g(a)$, $f(b) > g(b)$,

Then $\exists\ c \in (a,b)$ such that $f(c)=g(c)$

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Example: Does $\cos x - x = 0$ have a root?

Ans: Since $|\cos x| \le 1$

We have $\cos x - x > 0$ on $x = -2$

< 0 on $x = 2$

$\therefore \exists\ x_0 \in (-2, 2)$ s.t. $\cos x_0 - x_0 = 0$

Q: How to locate x_0 better?

Ans Bisection, see Section 2.4 for detail.