2.
14

$$\frac{d}{dx}\left(\int_x^{x^2}\sqrt{t^3+x^4}\,dt\right) \overset{④}{=} \frac{d}{dx}\left(\int_0^{x^2}\sqrt{t^3+x^4}\,dt\right) - \frac{d}{dx}\left(\int_0^{x}\sqrt{t^3+x^4}\,dt\right)$$

⑥
$$= \left(\sqrt{(x^2)^3+x^4}\cdot 2x + \int_0^{x^2}\frac{\partial}{\partial x}\sqrt{t^3+x^4}\,dt\right) - \left(\sqrt{x^3+x^4} + \int_0^{x}\frac{\partial}{\partial x}\sqrt{t^3+x^4}\,dt\right)$$

④
$$= \sqrt{x^6+x^4}\cdot 2x - \sqrt{x^3+x^4} + \int_x^{x^2}\frac{2x^3}{\sqrt{t^3+x^4}}\,dt.$$

4. $R = \{(x,y)\mid -2\le x\le 2, -2\le y\le 2\}$
14

(Sol-1) $\begin{cases} f_x = 2x+y = 0 \\ f_y = 2y+x = 0\end{cases}$ $\Rightarrow$ critical point : $(0,0)$. (in the interior of R)

② $f(0,0) = 0$.

Check boundaries : (corner : ④ , others ④)

$x=-2$: $f(x,y) = y^2-2y+4 \Rightarrow$ min. $= f(-2,1) = 3$, max $= f(-2,-2) = 12$

$x=2$: $f(x,y) = y^2+2y+4 \Rightarrow$ min. $= f(2,-1) = 3$, max. $= f(2,2) = 12$

By symmetry, $y=-2 \Rightarrow$ min. $= f(1,-2) = 3$, max. $= f(-2,-2) = 12$

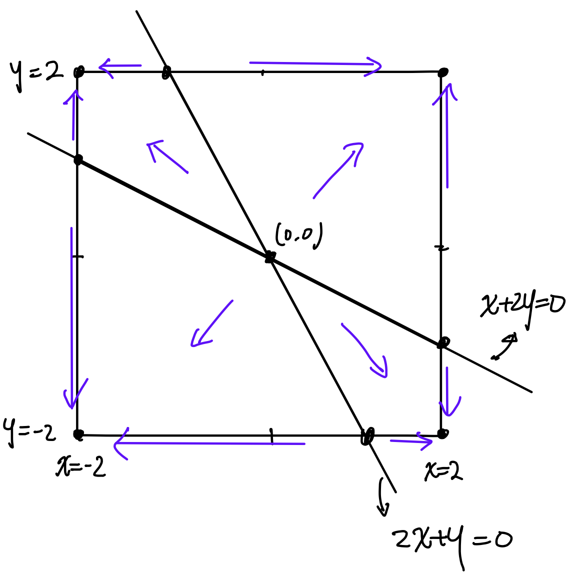
$y=2 \Rightarrow$ max $= f(-1,2) = 3$, max $= f(2,2) = 12$.

$\Rightarrow$ abs. min. $= 0 = f(0,0)$ ②, abs. max. $= 12 = f(2,2) = f(-2,-2)$. ②

(Sol-2). $\begin{cases} f_x = 2x+y = 0 \\ f_y = 2y+x = 0\end{cases}$ $\Rightarrow$ critical point : $(0,0)$. (in the interior of R)

② $f(0,0) = 0$.

Plot the gradient ⑥



$\Rightarrow$ abs. min. $= f(0,0) = 0$ ②

abs. max : check $f(\pm 2, \pm 2)$. ②

$f(2,2) = f(-2,-2) = 12$, $f(2,-2) = f(-2,2) = 4$

$\Rightarrow$ abs. max $= 12 = f(2,2) = f(-2,-2)$ ②

(sol-3). Note that $f(x,y) = \frac{1}{2}\left[(x+y)^2 + x^2 + y^2\right] \Rightarrow f(x,y) \geq 0$ (4)

, equality holds when $x = y = x+y = 0 \Leftrightarrow (x,y) = (0,0)$ $\therefore$ abs. min. $= f(0,0) = 0$.

(2)

To find max, note that the max. of x^2+y^2 for the points $(x,y) \in R$ occurs when (x,y) is the farthest to the origin. $\Rightarrow (x,y) = (\pm 2, \pm 2)$. (3)

Also, note that the max. of $|x+y|$ for the points $(x,y) \in R$ occurs when the line pass through (x,y) of slope -1 is the farthest to the origin $\Rightarrow (x,y) = (2,2)$ or $(-2,-2)$. (3)

$\Rightarrow$ abs. max. occurs when $(x,y) = (2,2)$ or $(-2,-2)$, value $= f(2,2) = 12$. (2)

5. $f(x,y,z) = xy + z^2$, $g(x,y,z) = y - x$, $h(x,y,z) = x^2+y^2+z^2-4$.

14

$\Rightarrow \nabla f = \langle y, x, 2z \rangle$, $\nabla g = \langle -1, 1, 0 \rangle$, $\nabla h = \langle 2x, 2y, 2z \rangle$. (2)

$$\Rightarrow \begin{cases} y = -\lambda + 2x\mu & \text{— ①} \\ x = \lambda + 2y\mu & \text{— ②} \\ 2z = 2z\mu & \text{— ③} \\ y - x = 0 & \text{— ④} \\ x^2+y^2+z^2 = 4 & \text{— ⑤} \end{cases}$$

(4)

By ③, $z = 0$ or $\mu = 1$

☆代入算 -4

If $z = 0 \xRightarrow{④⑤} x^2 = y^2 = 2$, $(x,y,z) = (\sqrt{2}, \sqrt{2}, 0), (-\sqrt{2}, -\sqrt{2}, 0)$ (3)

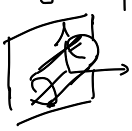
If $\mu = 1$, by ① − ② & ④, $\lambda = 0 \xRightarrow{①②} x = y = 0$.

$\xRightarrow{⑤} z = \pm 2 \Rightarrow (x,y,z) = (0, 0, \pm 2)$. (3)

Compare all points $\Rightarrow$ max. $= f(0,0,\pm 2) = 4$ $\longrightarrow$ sum = (2)

min. $= f(-\sqrt{2}, -\sqrt{2}, 0) = f(\sqrt{2}, \sqrt{2}, 0) = 2$

(Note: $\{(x,y,z) \mid y - x = 0,\ x^2+y^2+z^2 = 4\} = \{(x,y,z) \mid 2x^2 + z^2 = 4,\ y = x\}$

$\Rightarrow$  $\Rightarrow$ closed & bounded $\Rightarrow$ max & min. exist)

6. (sol-1). $f(x,y) = \frac{1}{1-x}\frac{1}{1-y}$ $\Rightarrow f_x = (1-x)^{-2}(1-y)^{-1}$, $f_y = (1-y)^{-2}(1-x)^{-1}$ (2)

14

$\Rightarrow f_{xx} = 2(1-x)^{-3}(1-y)^{-1}$, $f_{xy} = f_{yx} = (1-x)^{-2}(1-y)^{-2}$, $f_{yy} = 2(1-y)^{-3}(1-x)^{-1}$. (4)

$\Rightarrow f(0,0) = f_x(0,0) = f_y(0,0) = f_{xy}(0,0) = 1$, $f_{xx}(0,0) = f_{yy}(0,0) = 2$. (2)

$\Rightarrow f(x,y) \approx f(0,0) + (x f_x + y f_y)(0,0) + \frac{1}{2!}(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy})(0,0)$ (4)

$= 1 + (x+y) + (x^2 + xy + y^2)$ — (2)

#

6. (sol-2) Let $z = x+y-xy \Rightarrow f(x,y) = \frac{1}{1-z} = 1+z+z^2+\cdots$ (4)

$= 1+(x+y-xy)+(x^2+y^2+x^2y^2+2xy-2xy^2-2x^2y)+\cdots$ (5)

$\because$ quadratic $\therefore \approx 1+x+y-xy+x^2+y^2+2xy = 1+x+y+x^2+xy+y^2$ (5)

(sol-3) $f(x,y) = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right) = \left(1+x+x^2+R_2(x)\right)\left(1+y+y^2+R_2(y)\right)$ (4)

(5) Expand $= 1+x+y+x^2+xy+y^2+R_2(x,y)$ (degree >2 part)

with $\lim_{(x,y)\to(0,0)} \frac{R_2(x,y)}{x^2+y^2} = 0 \Rightarrow$ quadratic approx. $= 1+x+y+x^2+xy+y^2$ (5)

1.(a.) True +2 pt

f diff. at $(0,0)$ if $f_x(0,0)$ & $f_y(0,0)$ exist

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}} = 0 \quad \to +2\,pt$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \left[f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y\right] = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

+3pt

1.(b.) False +2 pt

Counterexample +2 pt

Explain this function is conti. but not diff. at $(0,0)$ +3pt

3.

$\nabla f = (yz, xz, xy) \Rightarrow \nabla f(1,1,1) = (1,1,1)$ → +4 pt

$\nabla g = (2x, 4y, 6z) \Rightarrow \nabla g(1,1,1) = (2,4,6)$ → +4 pt

法向量：$(1,1,1) \times (2,4,6) = (2,-4,2) \parallel (1,-2,1)$

↳ +3 pt

$\therefore \quad 1(x-1) - 2(y-1) + 1(z-1) = 0$

$\Rightarrow x - 2y + z = 0$ + 3 pt

7. (a.) Yes +2 pt

<法一> (Total 6 pt)

$$\lim_{(x,y)\to(0,0)} \left|\frac{x^2y^3}{x^4+y^4}\right| \overset{x=r\cos\theta}{\underset{y=r\sin\theta}{=\!=}} \lim_{r\to 0}\left|\frac{\cancel{r^5}^{\,r} \cos^2\theta\sin^3\theta}{\cancel{r^4}(\cos^4\theta+\sin^4\theta)}\right| \cdots (*)$$

+2 pt

Note by rmk.:

$$0 \le \left|\frac{\cos^2\theta\sin^3\theta}{\cos^4\theta+\sin^4\theta}\right| \le \frac{2|\cos^2\theta\sin^3\theta|}{(\cos^2\theta+\sin^2\theta)^2} \le 2$$

$$\therefore\ 0 \le (*) \le \lim_{r\to 0} r\cdot 2 = 0$$

By Squeeze thm., we have +2 pt

$$\lim_{r\to 0} r\frac{\cos^2\sin^3\theta}{\cos^4\theta\sin^4\theta} = 0 = f(0,0) \ \#$$

+2 pt

<法二> (Total 6 pt)

$$0 \le |f(x,y)| = \frac{|x^2y^3|}{x^4+y^4} \overset{\text{by rmk.}}{\le} \underline{\frac{2|x^2y^3|}{(x^2+y^2)^2}}$$ +2 pt

$$\le \frac{2(x^2+y^2)\cdot(x^2+y^2)^{3/2}}{(x^2+y^2)^2} = 2(x^2+y^2)^{\frac{1}{2}}$$

$\longrightarrow$ <u>0 as $(x,y)\to(0,0)$</u> +2pt

By squeeze thm., we have

$$\underline{\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(x,y)} \ \#$$ +2pt.

7. (b.) No +2pt

$f_x = f_y = 0$, zero function +2pt

$L(0,0) = f(0,0) = 0$ +2 pt

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}}$$ D.N.E. by two path thm.

+2 pt.