Midterm 1

Apr 07, 2026: 10:10AM

(I) All Theorems in the textbook can be used directly (unless you are asked precisely to prove that Theorem), just state clearly which Theorem you are using. Results proved in one problem can be used directly in another problem.

(II) All formula that you were asked to memorize, can be used directly unless you are asked explicitly to prove it.

1. (15 pts) Is the integral $\int_0^{\frac{\pi}{2}} \sqrt{\cot x}\, dx$ convergent? Do not try to find the anti-derivative.

2. (15 pts) Evaluate $\lim_{n\to\infty} \dfrac{\ln\left(\sum_{k=n}^{\infty} ke^{-k^2}\right)}{n^2}$. Give details.

3. (10 pts) Evaluate $\lim_{x\to 0} \dfrac{1}{x}\left(\dfrac{1}{\sin x} - \dfrac{1}{x}\right)$.

4. (15 pts) Use computational rules for power series (multiplication, differentiation, integration, etc.). to find <u>the value</u> of $\sum_{n=1}^{\infty} c_n x^n$ <u>and its radius of convergence</u>, where $c_n = n\left(\sum_{k=0}^{n} \dfrac{2^k}{(n-k)!}\right)$.

5. (10 pts) Give an approximation of $\int_0^{\frac{1}{3}} \cos(x^2)dx$ to within 10^{-8}. Give the formula of the approximation, but need not find the numerical value. Explain why the error is less than 10^{-6}.

6. (8+8+8 pts) True or False? Prove it if true, give a counter example if false.

   (a) If $\sum_{n=1}^{\infty} a_n x^n$ converges for $x = R$, then it converges absolutely on $|x| < R$.

   (b) If $\sum_{n=1}^{\infty} a_n x^n$ converges on $-R < x \le R$, then $\sum_{n=1}^{\infty} n^2 a_n x^n$ converges on $|x| < R$.

   (c) If $g(x) = f(0) + \sum_{n=1}^{\infty} \dfrac{f^{(n)}(0)}{n!} x^n$ converges on $|x| < 1$, then $f(x) = g(x)$ on $|x| < 1$.

7. (12 pts) Use any method to find $T_{\ln(1+x),0}(x)$. Then find the radius of convergence of this series.

1. $= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\tan x}}\,dx$. problem pt : $x=0$ 5

(15)

$\because \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{x}}\,dx$ conv. & $\lim_{x\to 0^+} \frac{1/\sqrt{\tan x}}{1/\sqrt{x}} = \left(\lim_{x\to 0^+} \frac{\tan x}{x}\right)^{\frac{1}{2}} = 1$. 5

Limit Comparison (for improper integral) $\Rightarrow$ ans = conv. 3 2

2. $\because f(x) = xe^{-x^2}$ decreasing for $x > N$ for some N 1 $\left(f'(x) = (1-2x^2)e^{-x^2} < 0 \text{ if } x > \frac{1}{\sqrt{2}} \right)$

(15)

& $\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} \overset{(L'H)}{=} \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0$. 1

$\therefore$ By the remainder estimation, $\int_n^\infty xe^{-x^2}dx < \sum_{k=n}^\infty ke^{-k^2} = S - S_{n-1} < \int_{n-1}^\infty xe^{-x^2}dx$. 5

$\int_n^\infty xe^{-x^2}dx = \cdots = \frac{1}{2}e^{-n^2}$. Similarly $\int_{n-1}^\infty$ " $= \frac{1}{2}e^{-(n-1)^2}$. 3

$\Rightarrow \frac{\ln\left(\int_n^\infty xe^{-x^2}dx\right)}{n^2} = -1 - \frac{\ln 2}{n^2} < \frac{\ln\left(\sum_{k=n}^\infty ke^{-k^2}\right)}{n^2} < -\left(\frac{n-1}{n}\right)^2 - \frac{\ln 2}{n^2} = \frac{\ln\left(\int_{n-1}^\infty \cdots\right)}{n^2}$

Left & right $\to -1$ as $n\to\infty$ 3 $\therefore$ ans. $= -1$. 2

3. $= \lim_{x\to 0} \frac{x - \sin x}{x^2 \sin x} = \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)}{x^3 - \frac{x^5}{3!} - \cdots} = \lim_{x\to 0} \frac{\frac{1}{6} - \frac{x^2}{5!} + \cdots}{1 - \frac{x^2}{6} + \cdots} = \frac{1}{6}$. 5 3 2

(10)

4. $\sum_{n=0}^\infty \Big(\underbrace{\sum_{k=0}^n 2^k \frac{1}{(n-k)!}}_{= a_n} \Big) x^n = \left(\sum_{n=0}^\infty 2^n x^n\right)\left(\sum_{n=0}^\infty \frac{1}{n!}x^n\right) = \frac{1}{1-2x}\cdot e^x$. 5 2

(15)

$f(x) = \frac{1}{1-2x}\cdot e^x = \sum_{n=0} a_n x^n \Rightarrow f'(x) = \sum_{n=1}^\infty n a_n x^{n-1} \Rightarrow x f'(x) = \sum_{n=1}^\infty n a_n x^n$. 3

$\Rightarrow \sum_{n=1}^\infty c_n x^n = \sum_{n=1}^\infty n a_n x^n = x\left(\frac{1}{1-2x}\cdot e^x\right)' = \frac{x(3-2x)e^x}{(2x-1)^2}$ #. 2

radius of conv.: $R\left(\frac{1}{1-2x}\right) = \frac{1}{2}$ $\left(|2x|<1 \Rightarrow |x| < \frac{1}{2}\right)$. $R(e^x) = \infty$

$\Rightarrow R(f(x)) = \frac{1}{2} \Rightarrow R(f'(x)) = \frac{1}{2} \Rightarrow R(xf'(x)) = \frac{1}{2}$ # 3

10 5. $\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots$

2pt

$$\therefore \int_0^{\frac{1}{3}} \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots\right) dx$$

$$= \left(x - \frac{x^5}{5 \cdot 2!} + \frac{x^9}{9 \cdot 4!} - \frac{x^{13}}{13 \cdot 6!} + \dots\right)\Big|_0^{\frac{1}{3}}$$

$$= \left(\frac{1}{3}\right) - \frac{\left(\frac{1}{3}\right)^5}{5 \cdot 2!} + \frac{\left(\frac{1}{3}\right)^9}{9 \cdot 4!} - \frac{\left(\frac{1}{3}\right)^{13}}{13 \cdot 6!} + \dots$$

Appr.

$$\Rightarrow \text{Appr.} = \sum_{n=0}^{2} (-1)^n \frac{\left(\frac{1}{3}\right)^{4n+1}}{(4n+1) \cdot (2n)!}$$

+4pt

$$|E| < \frac{\left(\frac{1}{3}\right)^{13}}{13 \cdot 6!} < 10^{-8}$$

4pt.

* 近似表字沒取好

+4

6. (a.) True 2pt

8

* 錯誤使用 ratio test 2pt

<pf> $\sum a_n R^n$ conv. $\Rightarrow a_n R^n \longrightarrow 0$ as $n \to \infty$

$\therefore \exists M > 0$ s.t. $|a_n| R^n \leq M$ 2pt

Note $|a_n x^n| = |a_n| R^n \cdot \left(\frac{|x|}{R}\right)^n \leq M \cdot \left(\frac{|x|}{R}\right)^n$ 2pt

Since $|x| < R \Rightarrow \frac{|x|}{R} < 1$

, So $\sum \left(\frac{|x|}{R}\right)^n$ conv. (Geo. Series)

$\Rightarrow a_n x^n$ conv. ab.

2pt

6^{8}. (b.) True 2pt

*錯誤使用 ratio test 2pt

〈法一〉

Note $\sum n^2 a_n x^n = \sum (a_n r^n) \cdot n^2 \left(\frac{x}{r}\right)^n$

where $|x| < r < R$ ↳ $(*)$ 2pt

Since x conv. at $x = R$, by 6.(a)

, we have $|a_n| r^n \le M$. for some

$M > 0$ → 2pt

So $(*) \le M \cdot n^2 \left|\frac{x}{r}\right|^n$

Similarly, we can have $n^2 \left|\frac{x}{r}\right|^n$

conv. by ratio test since $|x| < r$

$\Rightarrow \sum n^2 a_n x^n$ conv. 2pt.

〈法二〉 使用 term-by-term diff

State 微分不影響收斂半徑 4pt

$f = \sum a_n x^n$

$\Rightarrow \sum n^2 a_n x^n = x^2 f'' + x \cdot f' \cdots (**)$

$\because$ RHS of $(**)$ conv.

$\therefore$ LHS. of $(**)$ conv. 2pt

8

6.(c.) False 2pt ✓

Counterexample

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$ 3pt

$$f^{(n)}(x) = 0 \quad \forall\, n \geq 1$$ 3pt.

17

$*$ $T_{\ln(1+x),0}(x) = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \right.$

$\left. \text{or } = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \right)$

6pt

$*$ $R = 1$ 6pt

$*$ 如 計算錯誤 3pt.