Brief solutions to selected problems in homework week 14

1. Section 16.3: Solutions, common mistakes and corrections:

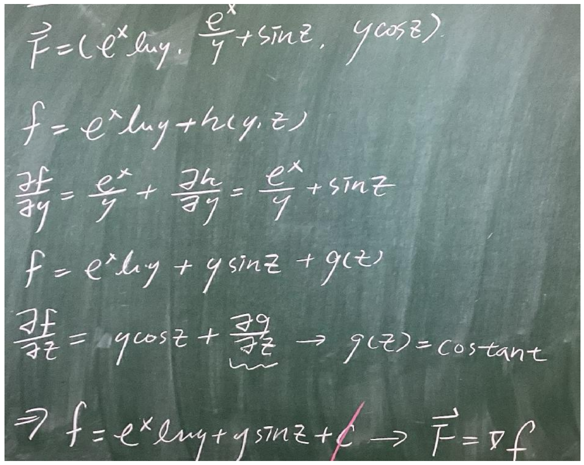

Figure 1: Solution to Section 16.3, problem 28

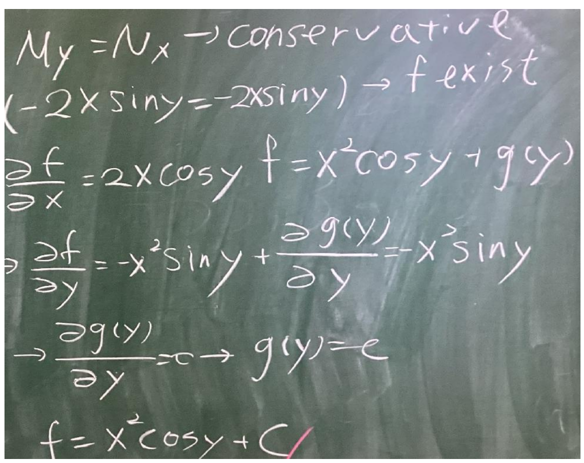

Figure 2: Solution to Section 16.3, problem 32(d)

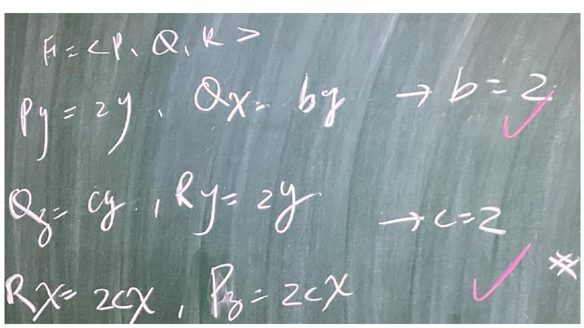

Figure 3: Solution to Section 16.3, problem 33(b). Answer = 0

2. Section 16.3: Problem 34: Read proof of Theorem 2, page 986-987.

3. Homework 14, problem 2.

   Problem 2 (a): Check by direct computation.

   Problem 2 (b): $\varphi(x,y,z) = \sqrt{x^2+y^2} + F(z)$ where $F'(z) = f(z)$ by observation.

   Problem 2 (c):

$$\oint_{x^2+y^2=1, z=0} \boldsymbol{G} \cdot \boldsymbol{T}\, ds = 2\pi \tag{1}$$

4. Problem 3: Suppose $\boldsymbol{F}$ satisfies the component test in $\{(x,y),\ x^2+y^2 \neq 0\}$. Let C be any simple closed curve, and Ω be the inside of C.

   (a) If $(0,0) \notin \Omega$.
   In this case, Ω is simply connected. We can apply the 2D version of 'Component Test for Conservative Field" statement on page 988, to conclude that ($\boldsymbol{F}$ is conservative, and therefore)

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = 0 \tag{2}$$

   (b) If $(0,0) \in \Omega$.
   As explained in Lecture 27, page 7 (see the figure there), we have

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \oint_{C_a} \boldsymbol{F} \cdot \boldsymbol{T}\, ds \tag{3}$$

   where $C_a = \{(x,y),\ x^2+y^2 = a^2\}$, $0 \ll a < 1$. Moreover, it is clear that the line integral in (3) is independent of $a > 0$.

   We conclude from the above analysis that,

   (i) If $\oint_{C_a} \boldsymbol{F} \cdot \boldsymbol{T}\, ds \neq 0$, then from Theorem 3 (loop property), $\boldsymbol{F}$ is not conservative.

   (ii) If $\oint_{C_a} \boldsymbol{F} \cdot \boldsymbol{T}\, ds = 0$, then we conclude from (2), (3) that

$$\oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = 0 \tag{4}$$

   for every simple closed curve C.

   (iii) If C is closed but not simple (i.e. C intersects itself), we can always decompose C into several simple closed curves (break up at the intersection points and reconnect), it follows that (4) remains valid even if C is closed but not simple.

   In summary, we have the following conclusion:

$$\boldsymbol{F} \text{ is conservative } \iff \oint_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = 0 \text{ for any closed curve } C \iff \oint_{C_a} \boldsymbol{F} \cdot \boldsymbol{T}\, ds = 0 \tag{5}$$

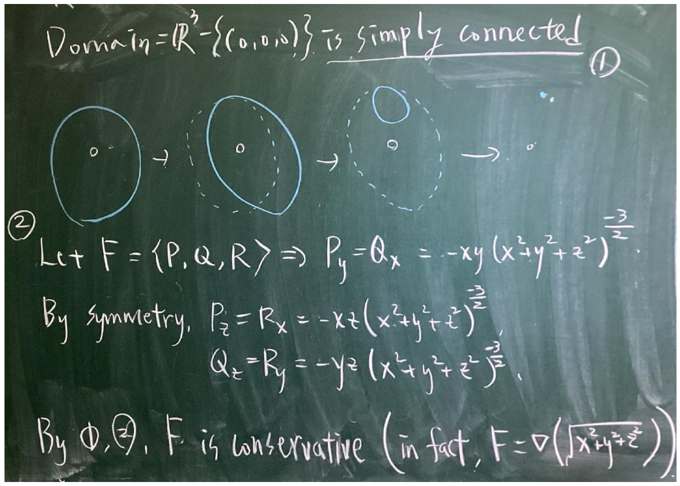

Figure 4: Solution to homework 14, problem 4

5. Section 16.4: Solutions, common mistakes and corrections:

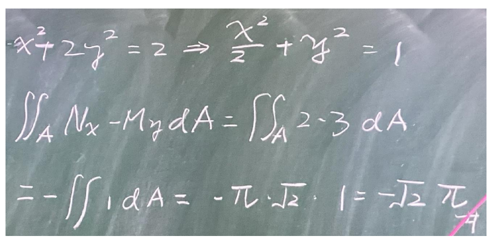

Figure 5: Solution to Section 16.4, problem 10

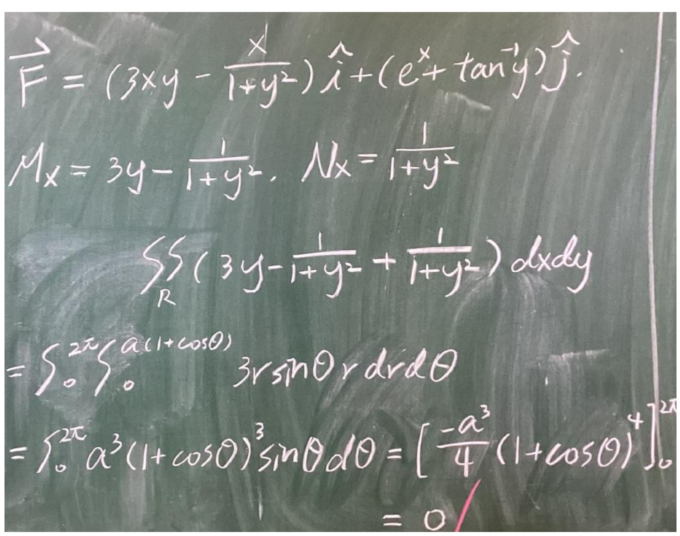

Figure 6: Solution to Section 16.4, problem 17

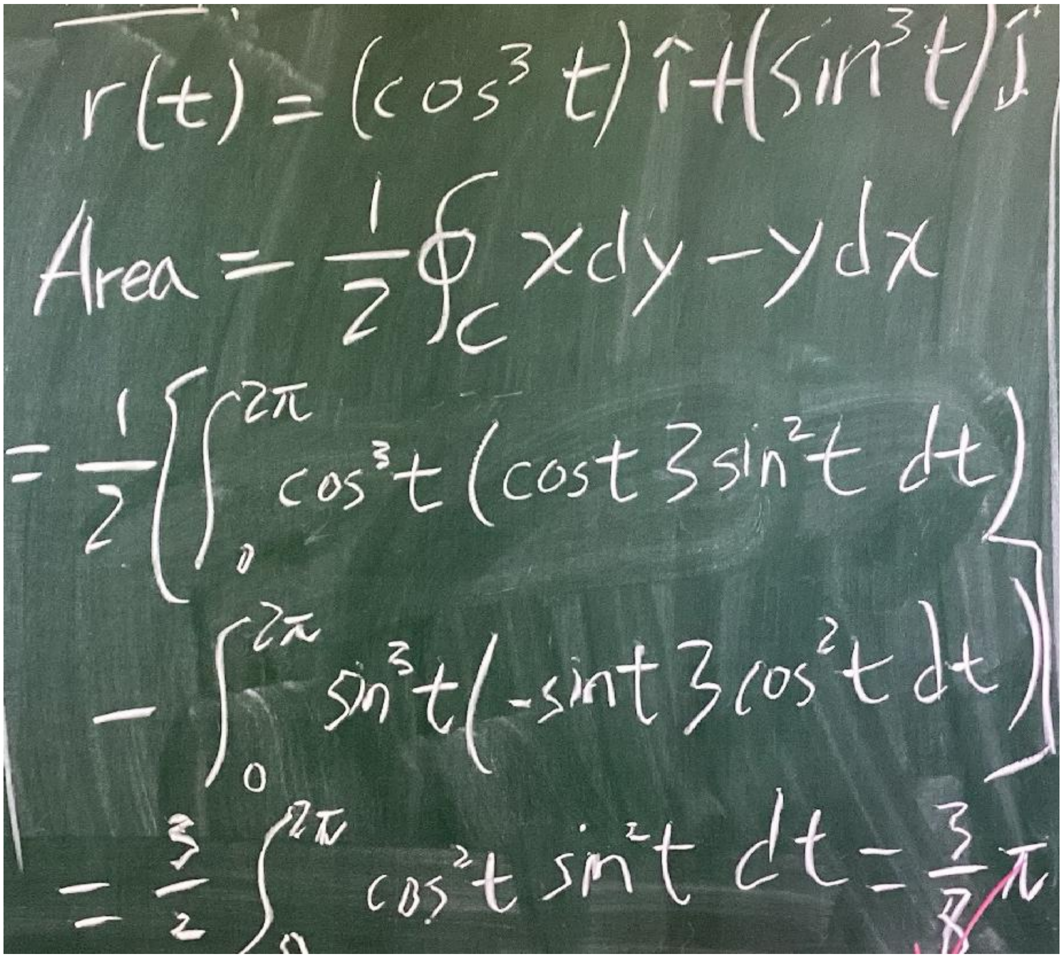

Figure 7: Solution to Section 16.4, problem 27

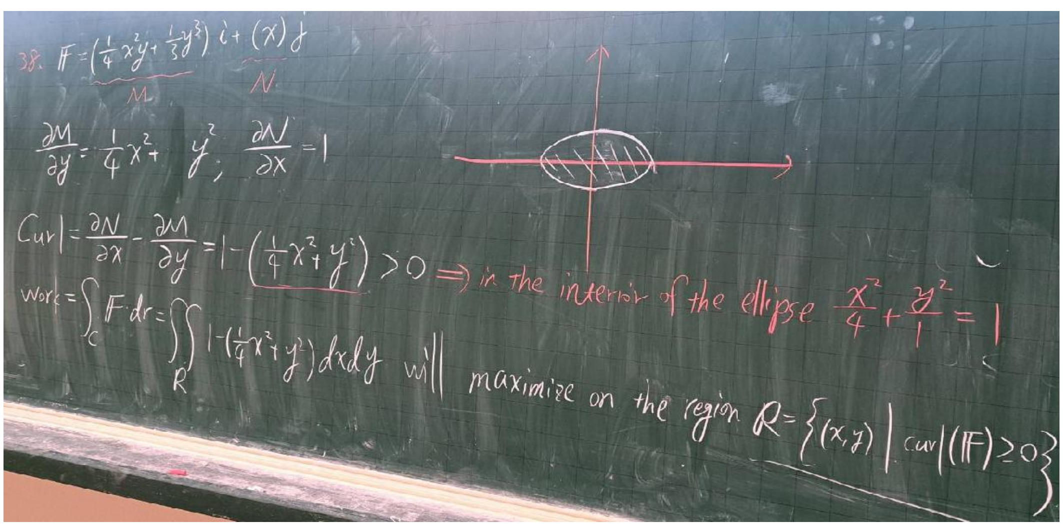

Figure 8: Solution to Section 16.4, problem 38

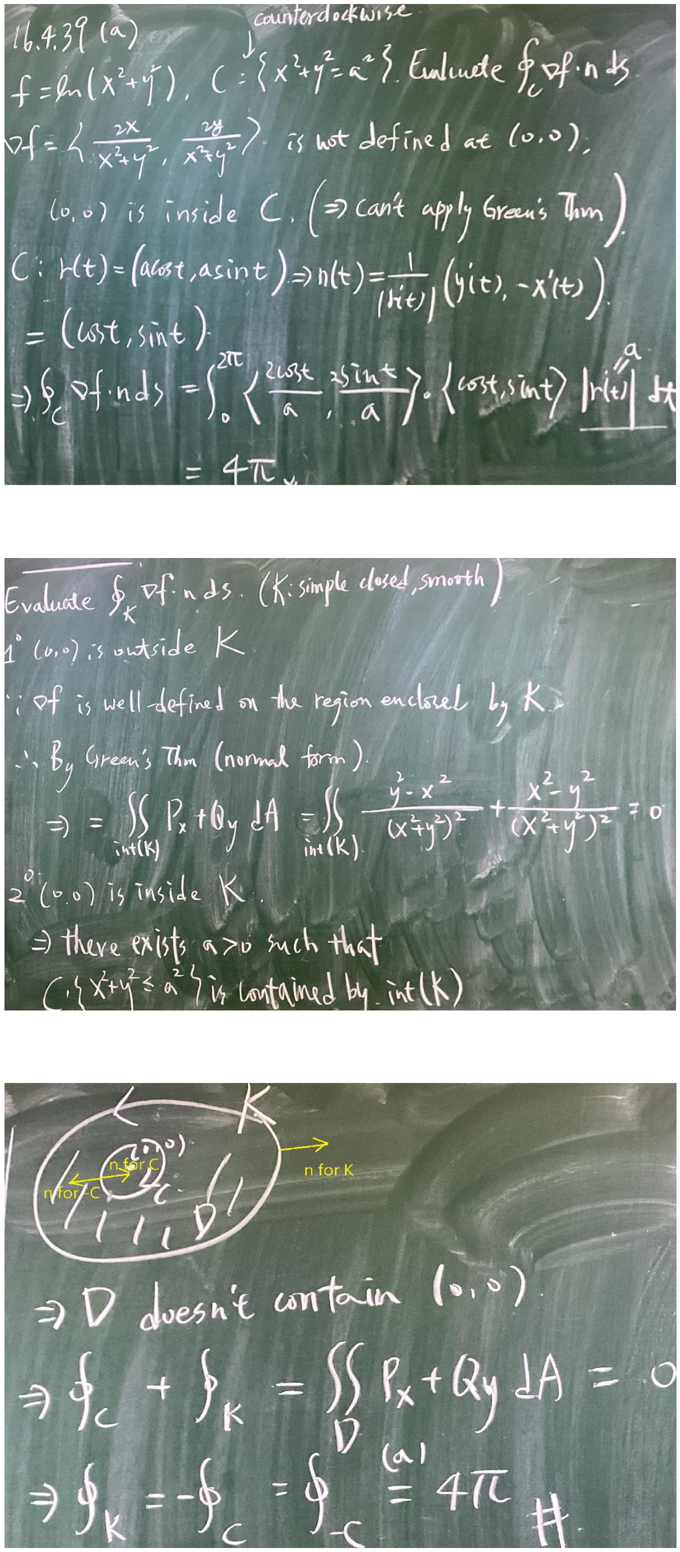

Figure 9: Solution to Section 16.4, problem 39. Remark on notation on last 2 lines: $\oint_K \nabla f\cdot \boldsymbol{n}\,ds + \oint_C \nabla f\cdot \boldsymbol{n}\,ds = 0$, $\oint_K \nabla f\cdot \boldsymbol{n}\,ds = -\oint_C \nabla f\cdot \boldsymbol{n}\,ds = \oint_C \nabla f\cdot \boldsymbol{n}\,ds = 4\pi$.