

Brief solutions to selected problems in homework week 14

1. Section 16.3: Solutions, common mistakes and corrections:

$$\vec{F} = (e^x \ln y, \frac{e^x}{y} + \sin z, y \cos z)$$

$$f = e^x \ln y + h(y, z)$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{y} + \frac{\partial h}{\partial y} = \frac{e^x}{y} + \sin z$$

$$f = e^x \ln y + y \sin z + g(z)$$

$$\frac{\partial f}{\partial z} = y \cos z + \frac{\partial g}{\partial z} \rightarrow g(z) = \text{constant}$$

$$\Rightarrow f = e^x \ln y + y \sin z + C \rightarrow \vec{F} = \nabla f$$

Figure 1: Solution to Section 16.3, problem 28

$$M_y = N_x \rightarrow \text{conservative}$$

$$(-2x \sin y = -2x \sin y) \rightarrow f \text{ exist}$$

$$\frac{\partial f}{\partial x} = 2x \cos y \quad f = x^2 \cos y + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin y + \frac{\partial g(y)}{\partial y} = -x^2 \sin y$$

$$\rightarrow \frac{\partial g(y)}{\partial y} = 0 \rightarrow g(y) = C$$

$$f = x^2 \cos y + C$$

Figure 2: Solution to Section 16.3, problem 32(d)

$$F = \langle P, Q, R \rangle$$

$$P_y = 2y, Q_x = by \rightarrow b = 2$$

$$Q_y = cy, R_y = 2y \rightarrow c = 2$$

$$R_x = 2cx, P_z = 2cx$$

Figure 3: Solution to Section 16.3, problem 33(b). Answer = 0

2. Section 16.3: Problem 34: Read proof of Theorem 2, page 986-987.
3. Homework 14, problem 2.

Problem 2 (a): Check by direct computation.

Problem 2 (b): $\varphi(x, y, z) = \sqrt{x^2 + y^2} + F(z)$ where $F'(z) = f(z)$ by observation.

Problem 2 (c):

$$\oint_{x^2+y^2=1, z=0} \mathbf{G} \cdot \mathbf{T} ds = 2\pi \quad (1)$$

4. Problem 3: Suppose \mathbf{F} satisfies the component test in $\{(x, y), x^2 + y^2 \neq 0\}$. Let C be any simple closed curve, and Ω be the inside of C .

- (a) If $(0, 0) \notin \Omega$.

In this case, Ω is simply connected. We can apply the 2D version of 'Component Test for Conservative Field' statement on page 988, to conclude that (\mathbf{F} is conservative, and therefore)

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = 0 \quad (2)$$

- (b) If $(0, 0) \in \Omega$.

As explained in Lecture 27, page 7 (see the figure there), we have

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = \oint_{C_a} \mathbf{F} \cdot \mathbf{T} ds \quad (3)$$

where $C_a = \{(x, y), x^2 + y^2 = a^2\}$, $0 \ll a < 1$. Moreover, it is clear that the line integral in (3) is independent of $a > 0$.

We conclude from the above analysis that,

- (i) If $\oint_{C_a} \mathbf{F} \cdot \mathbf{T} ds \neq 0$, then from Theorem 3 (loop property), \mathbf{F} is not conservative.
- (ii) If $\oint_{C_a} \mathbf{F} \cdot \mathbf{T} ds = 0$, then we conclude from (2), (3) that

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = 0 \quad (4)$$

for every simple closed curve C .

- (iii) If C is closed but not simple (i.e. C intersects itself), we can always decompose C into several simple closed curves (break up at the intersection points and reconnect), it follows that (4) remains valid even if C is closed but not simple.

In summary, we have the following conclusion:

$$\mathbf{F} \text{ is conservative} \iff \oint_C \mathbf{F} \cdot \mathbf{T} ds = 0 \text{ for any closed curve } C \iff \oint_{C_a} \mathbf{F} \cdot \mathbf{T} ds = 0 \quad (5)$$

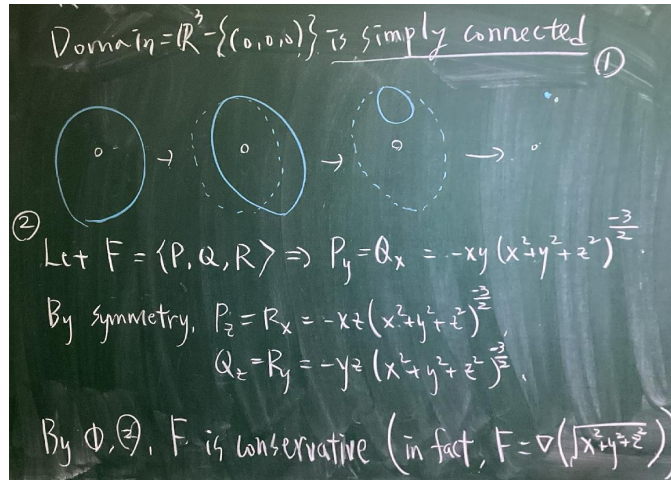


Figure 4: Solution to homework 14, problem 4

5. Section 16.4: Solutions, common mistakes and corrections:

$$x^2 + 2y^2 = 2 \Rightarrow \frac{x^2}{2} + y^2 = 1$$

$$\iint_A Nx - My \, dA = \iint_A 2 - 3 \, dA$$

$$= -\iint_A 1 \, dA = -\pi \cdot \sqrt{2} \cdot 1 = -\sqrt{2} \pi$$

Figure 5: Solution to Section 16.4, problem 10

$$\vec{F} = (3xy - \frac{x}{1+y^2})\hat{i} + (e^x + \tan^{-1}y)\hat{j}$$

$$M_x = 3y - \frac{1}{1+y^2}, \quad N_x = \frac{1}{1+y^2}$$

$$\iint_R (3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}) \, dx \, dy$$

$$= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} 3r \sin\theta \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} a^3 (1+\cos\theta)^3 \sin\theta \, d\theta = \left[\frac{-a^3}{4} (1+\cos\theta)^4 \right]_0^{2\pi}$$

$$= 0$$

Figure 6: Solution to Section 16.4, problem 17

$$\begin{aligned}
 \mathbf{r}(t) &= (\cos^3 t)\hat{i} + (\sin^3 t)\hat{j} \\
 \text{Area} &= \frac{1}{2} \oint_C x dy - y dx \\
 &= \frac{1}{2} \left[\int_0^{2\pi} \cos^3 t (\cos t \cdot 3 \sin^2 t dt) \right. \\
 &\quad \left. - \int_0^{2\pi} \sin^3 t (-\sin t \cdot 3 \cos^2 t dt) \right] \\
 &= \frac{3}{2} \int_0^{2\pi} \cos^2 t \sin^2 t dt = \frac{3}{8} \pi
 \end{aligned}$$

Figure 7: Solution to Section 16.4, problem 27

$$\text{ex. } F = \underbrace{\left(\frac{1}{4}xy + \frac{1}{3}y^3\right)}_M \hat{i} + \underbrace{(x)}_N \hat{j}$$

$$\frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2; \quad \frac{\partial N}{\partial x} = 1$$

$$\text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - \left(\frac{1}{4}x^2 + y^2\right) > 0 \Rightarrow \text{in the interior of the ellipse } \frac{x^2}{4} + \frac{y^2}{1} = 1$$

$$\text{work} = \int_C F \cdot dr = \iint_R \left(\frac{1}{4}x^2 + y^2\right) dx dy \text{ will maximize on the region } R = \{(x,y) \mid \text{curl}(F) \geq 0\}$$

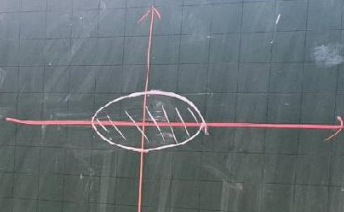
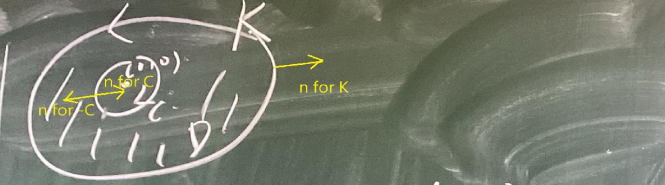


Figure 8: Solution to Section 16.4, problem 38

16.4.39 (a) ^{counterclockwise}
 $f = \ln(x^2 + y^2)$, $C = \{x^2 + y^2 = a^2\}$. Evaluate $\oint_C \nabla f \cdot \mathbf{n} \, ds$
 $\nabla f = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle$ is not defined at $(0,0)$,
 $(0,0)$ is inside C . (\Rightarrow can't apply Green's Thm.)
 $C: \mathbf{h}(t) = (a \cos t, a \sin t) \Rightarrow \mathbf{n}(t) = \frac{1}{|\mathbf{h}'(t)|} (y'(t), -x'(t))$
 $= (\cos t, \sin t)$
 $\Rightarrow \oint_C \nabla f \cdot \mathbf{n} \, ds = \int_0^{2\pi} \left\langle \frac{2a \cos t}{a}, \frac{2a \sin t}{a} \right\rangle \cdot \langle \cos t, \sin t \rangle | \mathbf{h}'(t) | \, dt$
 $= 4\pi$

Evaluate $\oint_K \nabla f \cdot \mathbf{n} \, ds$. (K : simple closed, smooth)
1. $(0,0)$ is outside K .
 $\therefore \nabla f$ is well-defined on the region enclosed by K .
 \therefore By Green's Thm (normal form).
 $\Rightarrow \iint_{\text{int}(K)} P_x + Q_y \, dA = \iint_{\text{int}(K)} \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dA = 0$
2. $(0,0)$ is inside K .
 \Rightarrow there exists $a > 0$ such that
 $C = \{x^2 + y^2 = a^2\}$ is contained by $\text{int}(K)$



$\Rightarrow D$ doesn't contain $(0,0)$.
 $\Rightarrow \oint_C + \oint_K = \iint_D P_x + Q_y \, dA = 0$
 $\Rightarrow \oint_K = -\oint_C = \oint_{-C} \stackrel{(a)}{=} 4\pi \neq$

Figure 9: Solution to Section 16.4, problem 39. Remark on notation on last 2 lines:

$$\oint_K \nabla f \cdot \mathbf{n} \, ds + \oint_C \nabla f \cdot \mathbf{n} \, ds = 0, \oint_K \nabla f \cdot \mathbf{n} \, ds = -\oint_C \nabla f \cdot \mathbf{n} \, ds = \oint_{-C} \nabla f \cdot \mathbf{n} \, ds = 4\pi.$$