

Brief solutions to selected problems in homework 12

1. Section 15.7: Solutions, common mistakes and corrections:

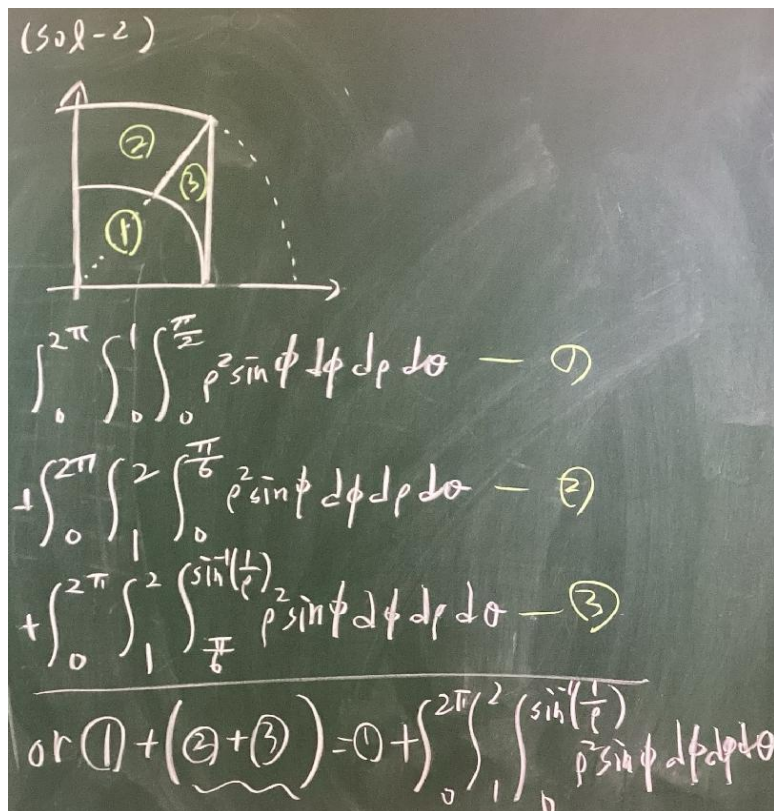
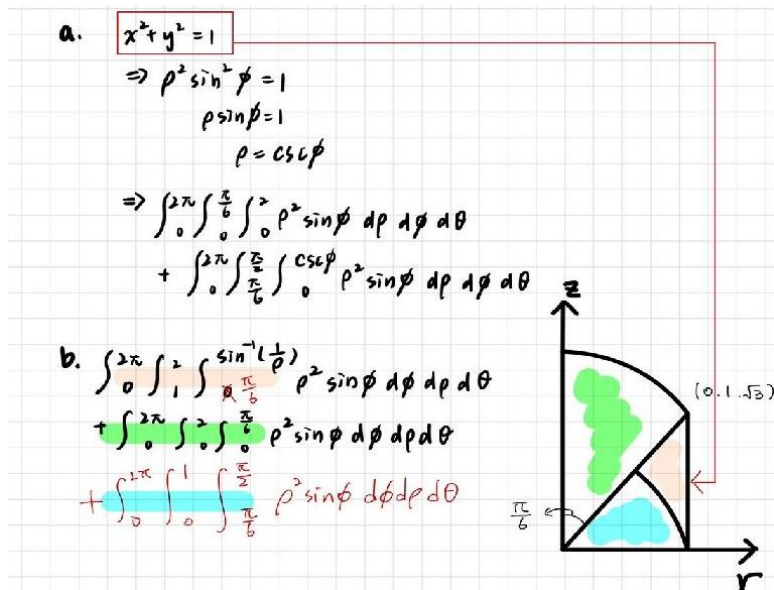


Figure 1: Solution to Section 15.7, problem 31

Cylindrical Surfaces for $\rho < 2$ in First Octant

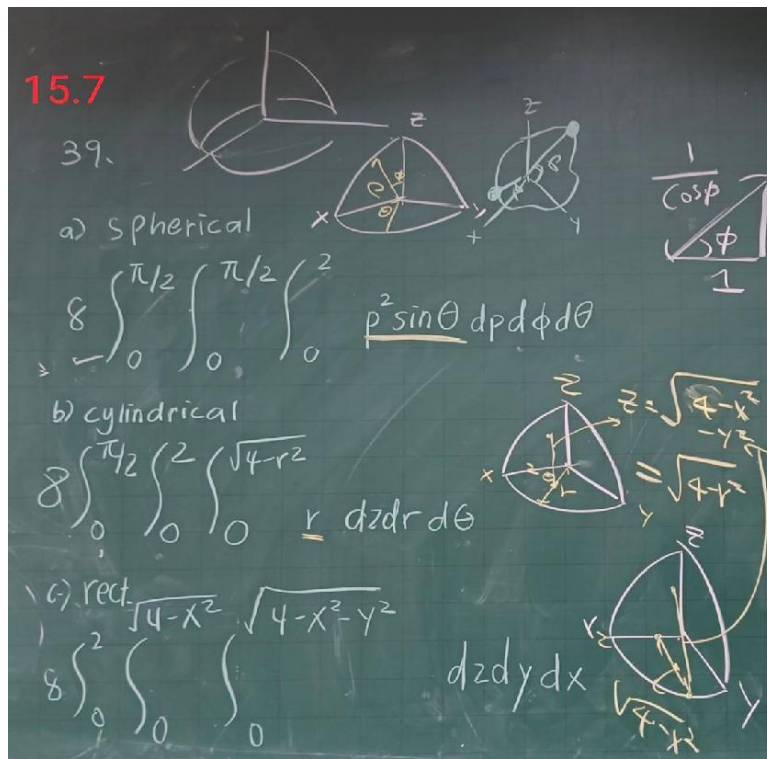
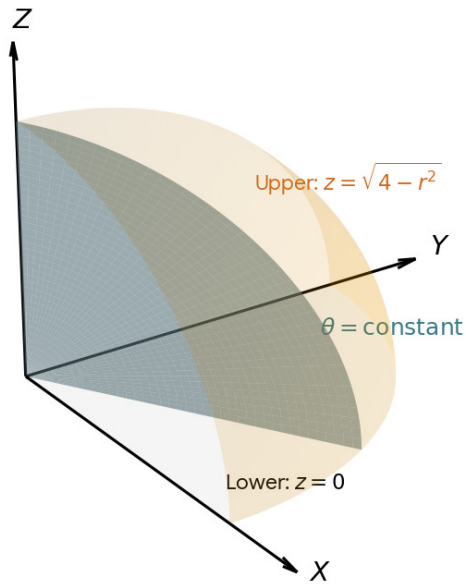
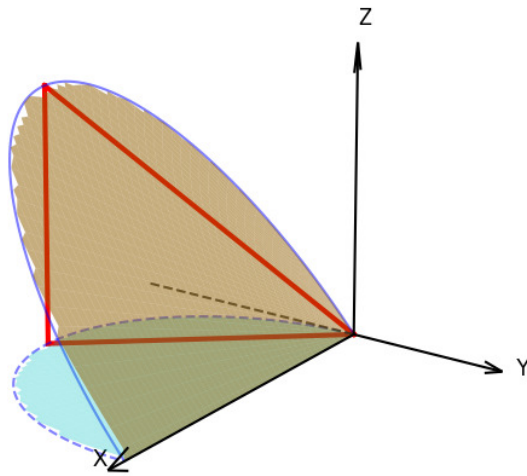


Figure 2: Solution to Section 15.7, problem 39



15.7

45. $z = -y$

$V = \int_{\frac{3\pi}{2}}^{2\pi} \int_0^{3\cos\theta} \int_0^{-r\sin\theta} dz r dr d\theta$

$= \int_{\frac{3\pi}{2}}^{2\pi} \int_0^{3\cos\theta} \left[z \Big|_0^{-r\sin\theta} \right] r dr d\theta$

$y = \int_{\frac{3\pi}{2}}^{2\pi} \int_0^{3\cos\theta} (-r\sin\theta) r dr d\theta$

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$= \int_{\frac{3\pi}{2}}^{2\pi} \int_0^{3\cos\theta} r^2 dr (-\sin\theta) d\theta$

$= \int_{\frac{3\pi}{2}}^{2\pi} \left[\frac{r^3}{3} \Big|_0^{3\cos\theta} \right] (-\sin\theta) d\theta$

$= \int_{\frac{3\pi}{2}}^{2\pi} \left[\frac{27\cos^3\theta}{3} \right] (-\sin\theta) d\theta$

Let $u = \cos\theta$

$du = -\sin\theta d\theta$

$= -9 \int_{\frac{3\pi}{2}}^{2\pi} (\cos^3\theta) (\sin\theta) d\theta$

$= 9 \int_0^1 u^3 du = 9 \left[\frac{u^4}{4} \right]_0^1$

$= \frac{9}{4}$

Figure 3: Solution to Section 15.7, problem 45

2. Section 15.8: Solutions, common mistakes and corrections:

15.8

7. $\iint_R 3x^2 + 4xy + 8y^2 \, dx \, dy$

$R: y = \frac{3}{2}x + 1, y = -\frac{x}{4} + 1, y = -\frac{3}{2}x + 3, y = -\frac{x}{4}$

$v = x + 4y$
 $u = 3x + 2y$

$\Rightarrow y = \frac{1}{10}(3v - u), x = \frac{1}{5}(2u - v)$

$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{10}$

For $y = -\frac{x}{4} + 1 \Rightarrow v = 0$
 For $y = \frac{3}{2}x + 1 \Rightarrow u = 2$
 For $y = -\frac{3}{2}x + 3 \Rightarrow u = 6$
 For $y = -\frac{x}{4} \Rightarrow v = 4$

$\int_2^6 \int_0^4 (uv) \frac{1}{10} \, dv \, du = \frac{64}{5}$

Figure 4: Solution to Section 15.8, problem 7

15.8

9. $\iint_R (\sqrt{\frac{y}{x}} + \sqrt{xy}) \, dx \, dy$ (bounded by $xy = 1, xy = 9, y = x, y = 4x, x = \frac{u}{v}, y = uv$)

$x = \frac{u}{v} \text{ \& } y = uv \Rightarrow \frac{y}{x} = v^2 \text{ \& } xy = u^2$

$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} v^{-1} & -uv^2 \\ v & u \end{vmatrix} = v^{-1}u + v^{-1}u = \frac{2u}{v}$

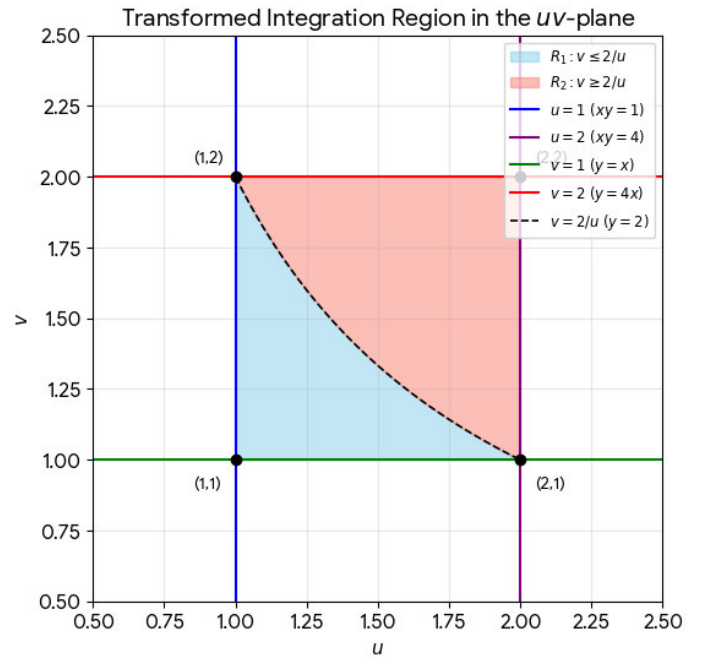
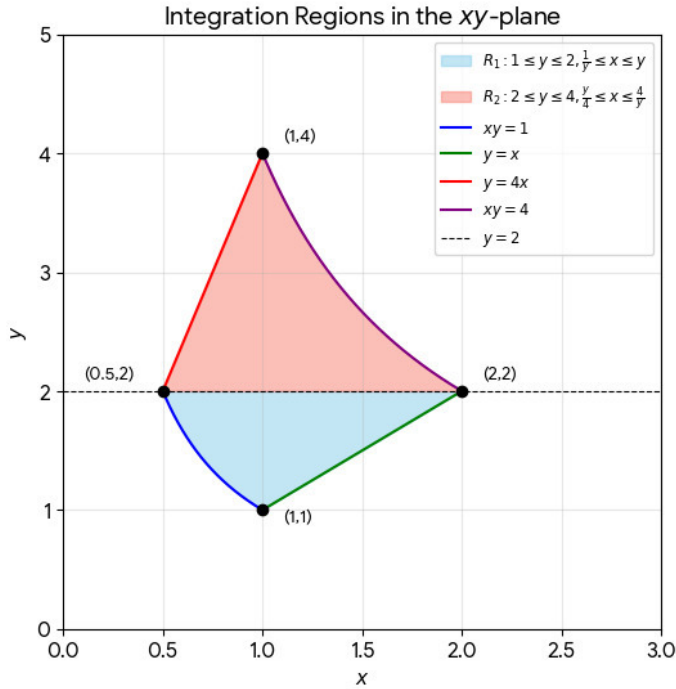
$y = x \Rightarrow \frac{u}{v} = uv \Rightarrow v = 1$; $y = 4x \Rightarrow \frac{4u}{v} = uv \Rightarrow v = 2$
 $xy = 1 \Rightarrow u = 1$; $xy = 9 \Rightarrow u = 3$

$\Rightarrow \int_1^3 \int_1^2 (v + u) \left(\frac{2u}{v}\right) \, dv \, du = \int_1^3 \int_1^2 \left(2u + \frac{2u^2}{v}\right) \, dv \, du$

$= \int_1^3 (2uv + 2u^2 \ln v) \Big|_1^2 \, du = \int_1^3 (2u + 2u^2 \ln 2) \, du$

$= u^2 + \frac{2}{3} u^3 \ln 2 \Big|_1^3 = 8 + \frac{52}{3} \ln 2 \#$

Figure 5: Solution to Section 15.8, problem 9



15.8.15

$x = \frac{u}{v}, y = uv$

• 第一部分積分 $1 \leq y \leq 2, \frac{1}{y} \leq x \leq y$
 • 第二部分積分 $2 \leq y \leq 4, \frac{y}{4} \leq x \leq \frac{4}{y}$

$x=y \Rightarrow \frac{u}{v} = uv, v = \pm 1$
 $4x=y \Rightarrow \frac{4u}{v} = uv, v = \pm 2$
 $xy=1 \Rightarrow u^2 = 1, u = \pm 1$
 $xy=4 \Rightarrow u^2 = 4, u = \pm 2$ (負不合, 因為 $x, y > 0$)

$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{2u}{v}$

$1+2 \Rightarrow [1, 2] \times [1, 2]$

$\int_1^2 \int_1^2 \left(\frac{u^2}{v^2} + u^2 v^2 \right) \frac{2u}{v} du dv = \int_1^2 \left[\frac{1}{2v^3} u^4 + \frac{v}{2} u^4 \right]_1^2 dv$

$= \int_1^2 \frac{15}{2v^3} + \frac{15v}{2} dv = \frac{225}{16} \checkmark$

Figure 7: Solution to Section 15.8, problem 15

15.8.19. Let $x=au, y=bv, z=cw$ with $a, b, c > 0$

$$\Rightarrow \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

and $|xyz| = abc |uvw|$ and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \Rightarrow u^2 + v^2 + w^2 \leq 1.$$

∴ $\text{PI} = \iiint_{\{u^2+v^2+w^2 \leq 1\}} abc |uvw| abc \, du \, dv \, dw$

$$= 8(abc)^2 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 (\rho \sin \phi \cos \theta) (\rho \sin \phi \sin \theta) \rho \cos \phi \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= 8(abc)^2 \left(\int_0^1 \rho^5 \, d\rho \right) \left(\int_0^{\frac{\pi}{2}} \sin^2 \phi \cos \phi \, d\phi \right) \left(\int_0^{\frac{\pi}{2}} \cos \theta \sin \theta \, d\theta \right)$$

$$= 8(abc)^2 \cdot \frac{1}{6} \cdot 1 \cdot 1$$

Figure 8: Solution to Section 15.8, problem 19

3. Chap 15: Solutions, common mistakes and corrections:

Chap 15. add.

11. $\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx \quad \left(\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-xy} dy \right)$

$$= \int_0^{\infty} \int_a^b e^{-xy} dy dx = \int_a^b \int_0^{\infty} e^{-xy} dx dy$$

$$= \int_a^b \lim_{c \rightarrow \infty} \int_0^c e^{-xy} dx dy = \int_a^b \lim_{c \rightarrow \infty} \left. -\frac{e^{-xy}}{y} \right|_0^c dy$$

$$= \int_a^b \frac{1}{y} dy = \ln y \Big|_a^b$$

$$= \ln b - \ln a$$

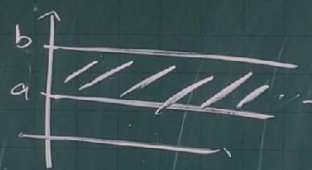
$$= \ln \frac{b}{a} \#$$


Figure 9: Solution to Chapter 15, Additional and Advanced Exercises, problem 11

S.15. Additional 12

(a) $y \cot \beta \leq x \leq \sqrt{a^2 - y^2}$ $0 \leq y \leq a \sin \beta$
 $x^2 + y^2 = a^2$ $\frac{y}{x} = \tan \beta$

$\int_0^{a \sin \beta} \int_{y \cot \beta}^{\sqrt{a^2 - y^2}} \ln(x^2 + y^2) dx dy = \int_0^\beta \int_0^a (2 \ln r) \cdot r dr d\theta$

(b) $x=0 \sim x = a \cos \beta$ $\Rightarrow a^2 \beta (\ln a - \frac{1}{2})$
 $y=0$ $x = y \cot \beta \rightarrow y = x \tan \beta$

① $\int_0^{a \cos \beta} \int_0^{x \tan \beta} \ln(x^2 + y^2) dy dx$ $\int_0^{a \cos \beta} \int_0^{x \tan \beta} \ln(x^2 + y^2) dy dx$

$x = a \cos \beta \sim x = a$
 $y=0$ $x^2 + y^2 = a^2$ $y = \sqrt{a^2 - x^2}$ if $\int_{a \cos \beta}^a \int_0^{\sqrt{a^2 - x^2}} \ln(x^2 + y^2) dy dx$

$\int_0^a \ln(r^2) r dr = \int_0^a \ln(u) \frac{1}{2} du$ (let $u = r^2$, $du = 2r dr$)
 $= \frac{1}{2} [u \ln(u) - u]_0^a$ (integration by parts)
 $= \frac{1}{2} (a^2 \ln(a^2) - a^2) - \frac{1}{2} \lim_{u \rightarrow 0^+} (u \ln(u) - u)$
 $= a^2 \ln(a) - \frac{a^2}{2}$ (L'Hôpital rule)

Figure 10: Solution to Chapter 15, Additional and Advanced Exercises, problem 12

S.15. Practice 54.

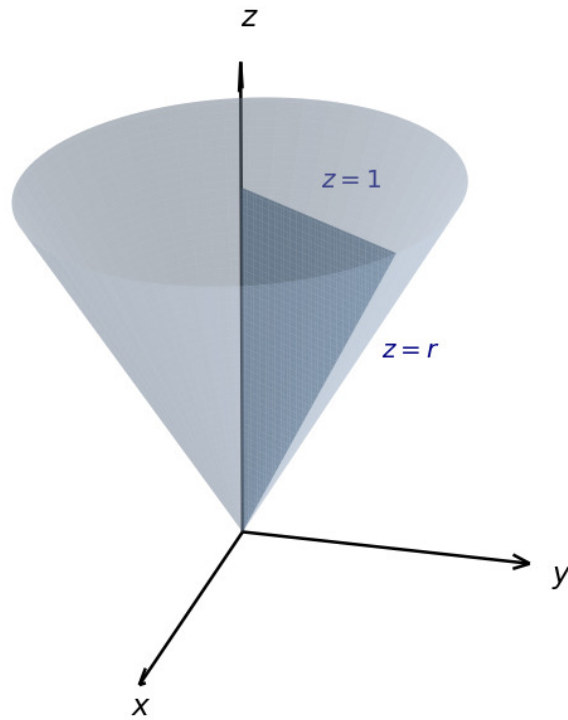
(Assume there exists $\alpha, \beta, \gamma, \delta$ such that $s = \alpha x + \beta y$
 $t = \gamma x + \delta y$
 and $\alpha x^2 + 2\beta \gamma xy + \gamma y^2 = s^2 + t^2$ and $\alpha \beta - \gamma \delta = ac - b^2$)

$\frac{\partial(s, t)}{\partial(x, y)} = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \sqrt{ac - b^2}$

$\iint_G e^{-(s^2 + t^2)} \frac{1}{\sqrt{ac - b^2}} ds dt = \frac{\pi}{\sqrt{ac - b^2}} = 1 \Rightarrow \underline{ac - b^2 = \pi^2}$ $a > 0$

$\iint_G e^{-(s^2 + t^2)} ds dt = \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta = \pi$ (15.4.4)

Figure 11: Chapter 15, Practice Exercises, problem 54. Sorry, this problem is not stated properly. My mistake. Please ignore it in all exams. I will update with a more complete answer some time later



$$\begin{aligned}
 & \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^1 dz \, dy \, dx \\
 &= \int_0^{2\pi} \int_0^1 \int_r^1 r \, dz \, dr \, d\theta \\
 &= 2\pi \int_0^1 r(1-r) \, dr \\
 &= \frac{\pi}{3} \checkmark
 \end{aligned}$$

Figure 12: Solution to Chapter 15, Practice Exercises, problem 33