

Brief solutions to selected problems in homework 11

1. Section 15.4: Solutions, common mistakes and corrections:

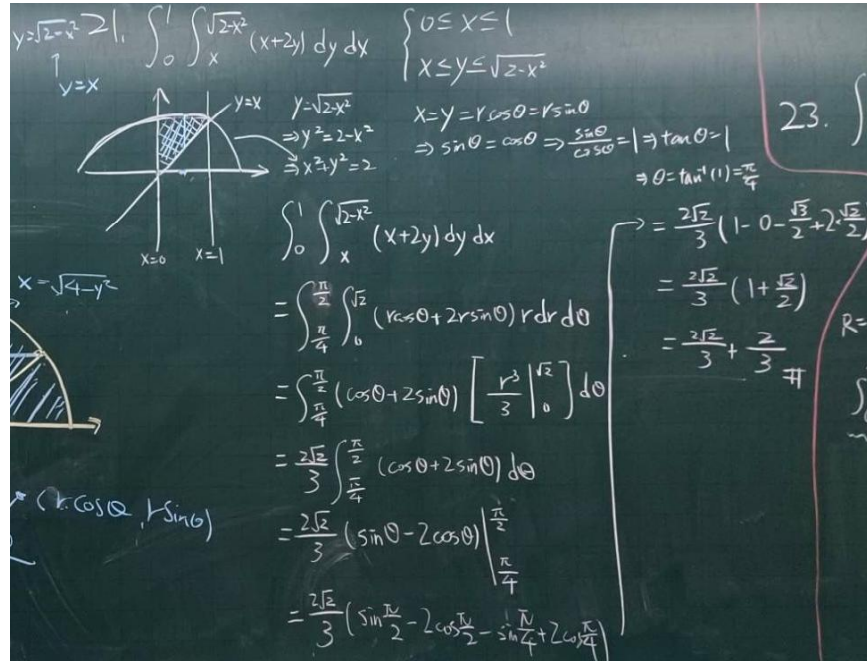


Figure 1: Solution to Section 15.4, problem 21

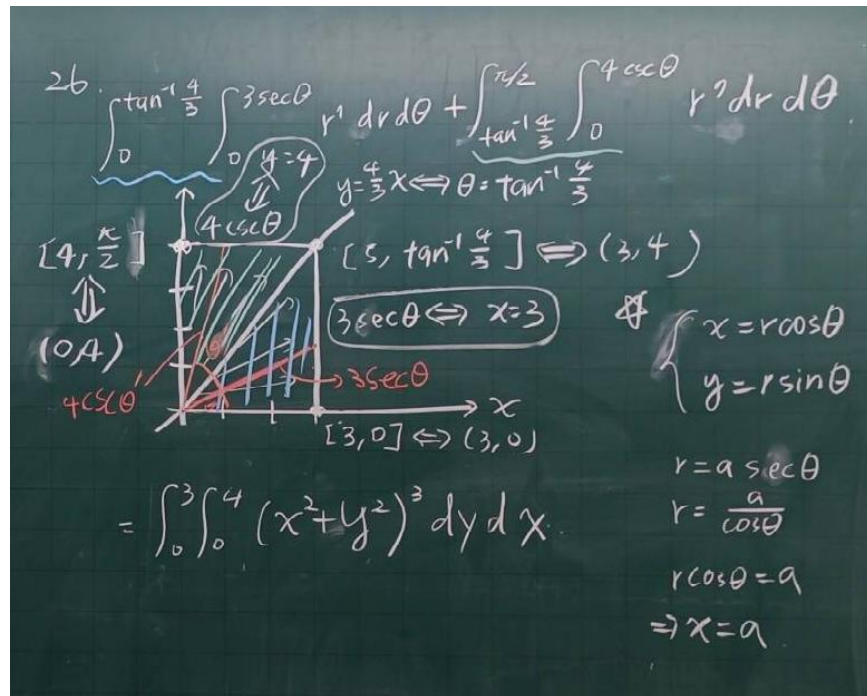


Figure 2: Solution to Section 15.4, problem 26

15.4.41 (a)

$$I^2 = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy \quad x^2+y^2=r^2 \quad 0 < \theta < \frac{\pi}{2}$$

$$0 \leq r \leq \infty \Rightarrow \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} r dr d\theta$$

$$= \frac{\pi}{4} \int_0^{\infty} e^{-u} du = \frac{\pi}{4} \quad \text{Since } I > 0 \quad I = \frac{\sqrt{\pi}}{2} \checkmark$$

(a = r²) (b)

$$\lim_{x \rightarrow \infty} \operatorname{erf}(x) = \lim_{x \rightarrow \infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}} dt$$

$$= \lim_{x \rightarrow \infty} \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \quad \text{from (a)} \quad I = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \frac{2}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} = 1 \quad \checkmark$$

Figure 3: Solution to Section 15.4, problem 41

$$\int_0^{\infty} \int_0^{\infty} \frac{1}{(1+x^2+y^2)^2} dx dy$$

$$= \int_0^{\pi/2} \int_0^{\infty} \frac{r}{(1+r^2)^2} dr d\theta \quad (\because x = r \cos \theta, y = r \sin \theta)$$

$$= \frac{\pi}{2} \lim_{b \rightarrow \infty} \int_0^b \frac{r}{(1+r^2)^2} dr$$

$$= \frac{\pi}{4} \lim_{b \rightarrow \infty} \left[1 - \frac{1}{1+r^2} \right]_0^b$$

$$= \frac{\pi}{4} \lim_{b \rightarrow \infty} \left(1 - \frac{1}{1+b^2} \right) = \frac{\pi}{4} \quad \checkmark$$

Figure 4: Solution to Section 15.4, problem 42

2. Section 15.5: Solutions, common mistakes and corrections:

HW13. Q3.

Q: Find $dV = dx dy dz$

① for x : $\begin{cases} \text{左: } x=0 \\ \text{右: } x-y+z=0 \Rightarrow x=y-z \end{cases}$
 $\Rightarrow \int_0^{y-z} dx$

② for y : $\begin{cases} \text{左: } z-y=0 \Rightarrow y=z \\ \text{右: } y=1 \end{cases}$
 $\Rightarrow \int_z^1 \int_0^{y-z} dx dy$

③ for z : $\begin{cases} \text{上: } z=1 \\ \text{下: } z=0 \end{cases}$
 $\Rightarrow \int_0^1 \int_z^1 \int_0^{y-z} dx dy dz$

calculate:
 $\int_0^1 \int_z^1 \int_0^{y-z} dx dy dz$
 $= \int_0^1 \int_z^1 (y-z-0) dy dz$
 $= \int_0^1 \left[\frac{y^2}{2} - zy \right]_{y=z}^{y=1} dz$
 $= \int_0^1 \left[\left(\frac{1}{2} - z \right) - \left(\frac{z^2}{2} - z^2 \right) \right] dz$
 $= \int_0^1 \left(\frac{z^2}{2} - z + \frac{1}{2} \right) dz$
 $= \left. \frac{z^3}{6} - \frac{z^2}{2} + \frac{z}{2} \right|_{z=0}^{z=1}$
 $= \frac{1}{6} \#$

Figure 5: Solution to homework 11, problem 2

(a) $\int_{-1}^1 \int_0^{1-x^2} \int_{x^2}^{1-z} dy dz dx$ ✓
 $(-1 \leq x \leq 1, 0 \leq z \leq 1-x^2, x^2 \leq y \leq 1-z)$

(b) $\int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_{x^2}^{1-z} dy dx dz$ ✓
 $(0 \leq z \leq 1, -\sqrt{1-z} \leq x \leq \sqrt{1-z}, x^2 \leq y \leq 1-z)$

(c) $\int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx dy dz$ ✓
 $(0 \leq z \leq 1, 0 \leq y \leq 1-z, -\sqrt{y} \leq x \leq \sqrt{y})$

Figure 6: Solution to Section 15.5, problem 21 (a), (b), (c)

(d) $-\sqrt{y} \leq x \leq \sqrt{y}$
 $0 \leq z \leq 1-y$
 $0 \leq y \leq 1$
 $\int_0^1 \int_0^{1-y} \int_{-\sqrt{y}}^{\sqrt{y}} dz dx dy$ ✓

(e) $0 \leq z \leq 1-y$
 $-\sqrt{y} \leq x \leq \sqrt{y}$
 $0 \leq y \leq 1$
 $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^{1-y} dz dx dy$ ✓

Figure 7: Solution to Section 15.5, problem 21 (d), (e)

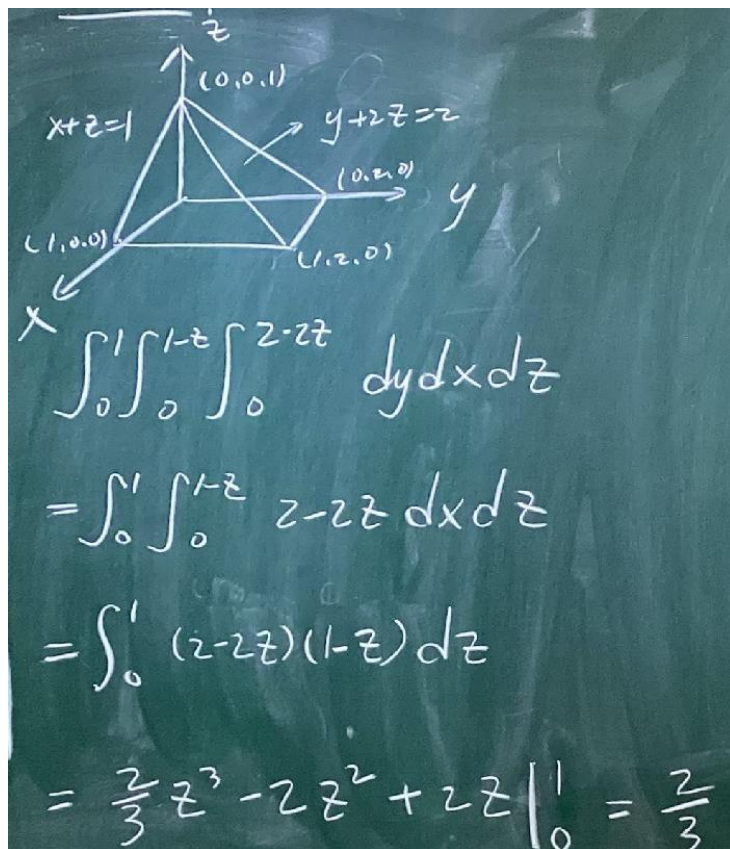


Figure 8: Solution to Section 15.5, problem 24

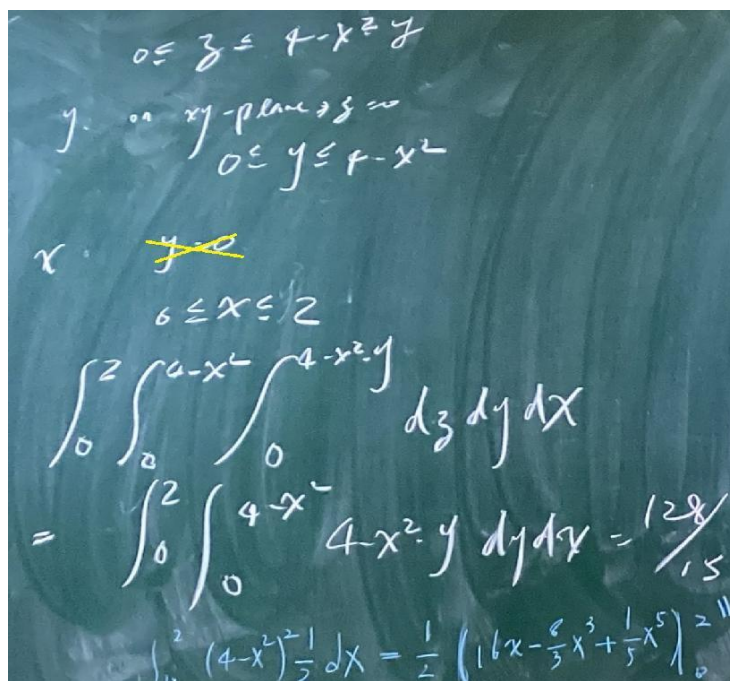


Figure 9: Solution to Section 15.5, problem 30

$$\begin{aligned}
& \int_{-1}^1 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{x+2} dz dx dy \\
&= \int_{-1}^1 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x+2) dx dy \\
&= \int_{-1}^1 \left(\frac{x^2}{2} + 2x \Big|_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \right) dy \\
&= \int_{-1}^1 4\sqrt{4-y^2} dy = \int_{-1}^1 8\sqrt{1-y^2} dy \\
&= 8 \times \frac{1}{2} \times \pi \times 1^2 = 4\pi
\end{aligned}$$

Figure 10: Solution to Section 15.5, problem 35

$$\begin{aligned}
& \int_0^4 \int_0^1 \int_{2y}^x \frac{2\cos(x^2)}{\sqrt{z}} dz dy dx \\
& 2y < x \Rightarrow y < \frac{x}{2} \\
& 0 \leq y \leq 1 \quad 0 \leq x \leq 2 \\
& \Rightarrow 0 \leq x \leq 2 \quad 0 \leq y \leq \frac{x}{2} \\
& \int_0^4 \int_0^2 \int_0^{\frac{x}{2}} \frac{2\cos(x^2)}{\sqrt{z}} dy dx dz \\
&= \int_0^4 \int_0^2 \frac{x\cos(x^2)}{\sqrt{z}} dx dz \\
&= \left(\int_0^4 \frac{1}{\sqrt{z}} dz \right) \left(\int_0^2 x\cos(x^2) dx \right) = 4 \cdot \frac{1}{2} \sin 4 \\
&= 2\sin 4
\end{aligned}$$

Figure 11: Solution to Section 15.5, problem 41

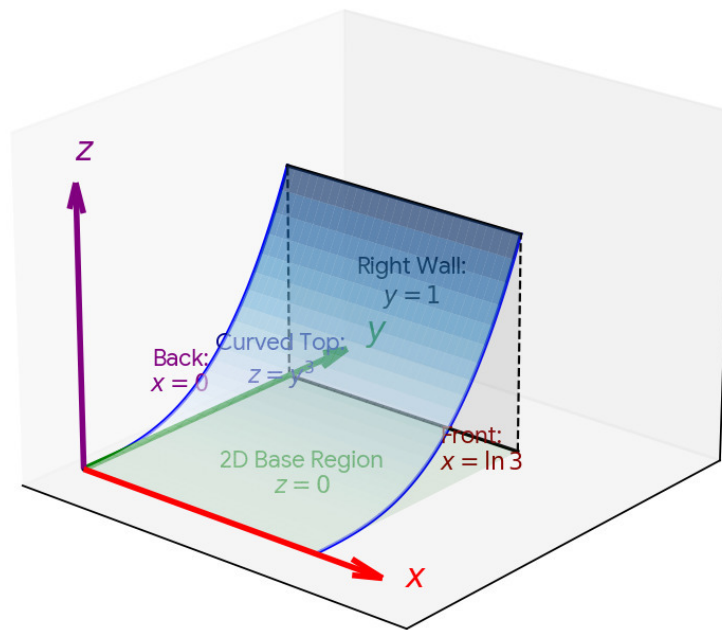
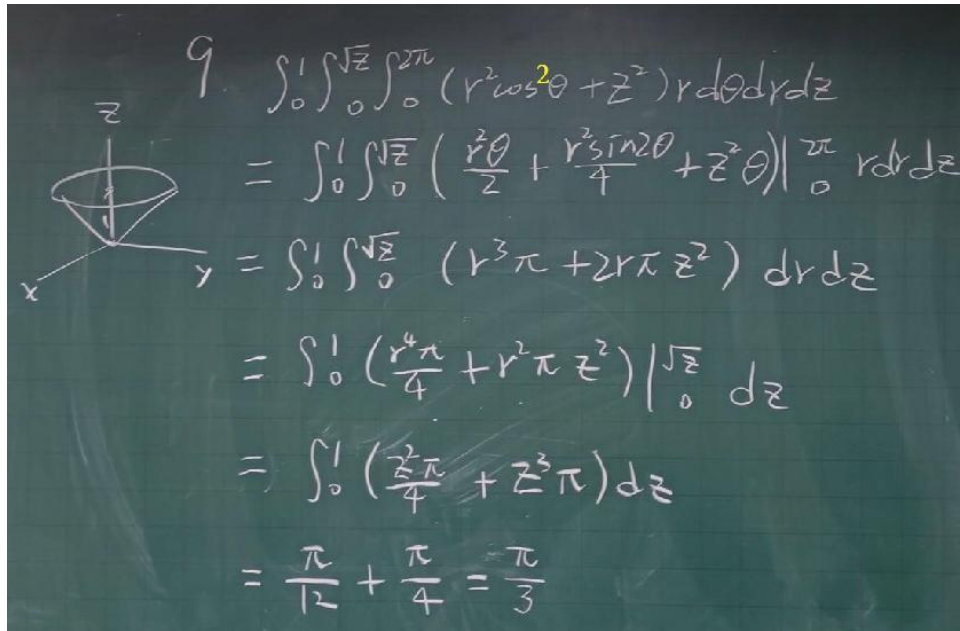


Figure 12: Solution to Section 15.5, problem 43

$$\begin{aligned}
 & \int_0^1 \int_{\sqrt{z}}^1 \int_0^{\ln 3} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} dx dy dz \\
 &= \int_{x=0}^{\ln 3} \int_{y=0}^1 \int_{z=0}^{y^2} \frac{\pi e^{2x} \sin(\pi y^2)}{y^2} dz dy dx \\
 &= \int_{x=0}^{\ln 3} \int_{y=0}^1 \pi e^{2x} y \sin(\pi y^2) dy dx \\
 &= \int_{x=0}^{\ln 3} \left. -\frac{e^{2x}}{2} \cos(\pi y^2) \right|_0^1 dx \\
 &= \int_{x=0}^{\ln 3} e^{2x} dx \\
 &= \left. \frac{e^{2x}}{2} \right|_0^{\ln 3} = 4 \quad \#
 \end{aligned}$$

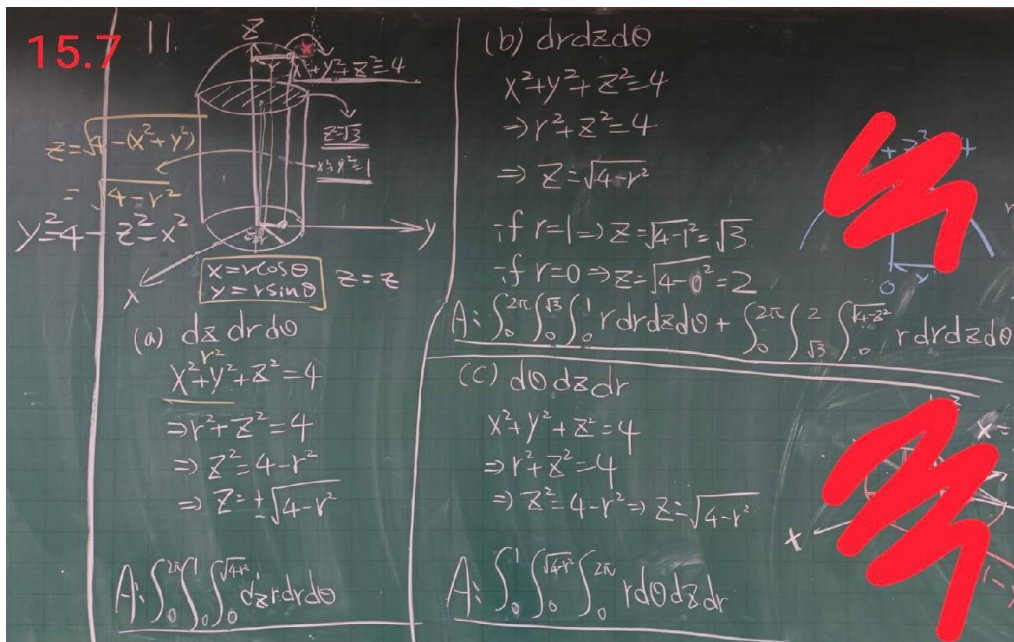
Figure 13: Solution to Section 15.5, problem 43. First plot the $x = \text{constant}$ cross section, then change $dydz$ to $dzdy$ on this cross section

3. Section 15.7: Solutions, common mistakes and corrections:



9. $\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2 \theta + z^2) r d\theta dr dz$
 $= \int_0^1 \int_0^{\sqrt{z}} \left(\frac{r^2 \theta}{2} + \frac{r^2 \sin^2 \theta}{4} + z^2 \theta \right) \Big|_0^{2\pi} r dr dz$
 $= \int_0^1 \int_0^{\sqrt{z}} (r^3 \pi + 2r\pi z^2) dr dz$
 $= \int_0^1 \left(\frac{r^4 \pi}{4} + r^2 \pi z^2 \right) \Big|_0^{\sqrt{z}} dz$
 $= \int_0^1 \left(\frac{z^2 \pi}{4} + z^3 \pi \right) dz$
 $= \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$

Figure 14: Solution to Section 15.7, problem 9



15.7 11. $x^2 + y^2 + z^2 = 4$
 $z = \sqrt{4 - x^2 - y^2}$
 $x = r \cos \theta$
 $y = r \sin \theta$
 $z = z$
 (a) $dz dr d\theta$
 $x^2 + y^2 + z^2 = 4$
 $\Rightarrow r^2 + z^2 = 4$
 $\Rightarrow z^2 = 4 - r^2$
 $\Rightarrow z = \pm \sqrt{4 - r^2}$
 $A: \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} \int_0^{\sqrt{4-r^2}} dz r dr d\theta$
 (b) $dr dz d\theta$
 $x^2 + y^2 + z^2 = 4$
 $\Rightarrow r^2 + z^2 = 4$
 $\Rightarrow z = \sqrt{4 - r^2}$
 if $r=1 \Rightarrow z = \sqrt{4-1} = \sqrt{3}$
 if $r=0 \Rightarrow z = \sqrt{4-0} = 2$
 $A: \int_0^{2\pi} \int_0^{\sqrt{3}} \int_0^1 r dr dz d\theta + \int_0^{2\pi} \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-z^2}} r dr dz d\theta$
 (c) $d\theta dz dr$
 $x^2 + y^2 + z^2 = 4$
 $\Rightarrow r^2 + z^2 = 4$
 $\Rightarrow z^2 = 4 - r^2 \Rightarrow z = \sqrt{4 - r^2}$
 $A: \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} \int_0^{\sqrt{4-r^2}} r d\theta dz dr$

Figure 15: Solution to Section 15.7, problem 11

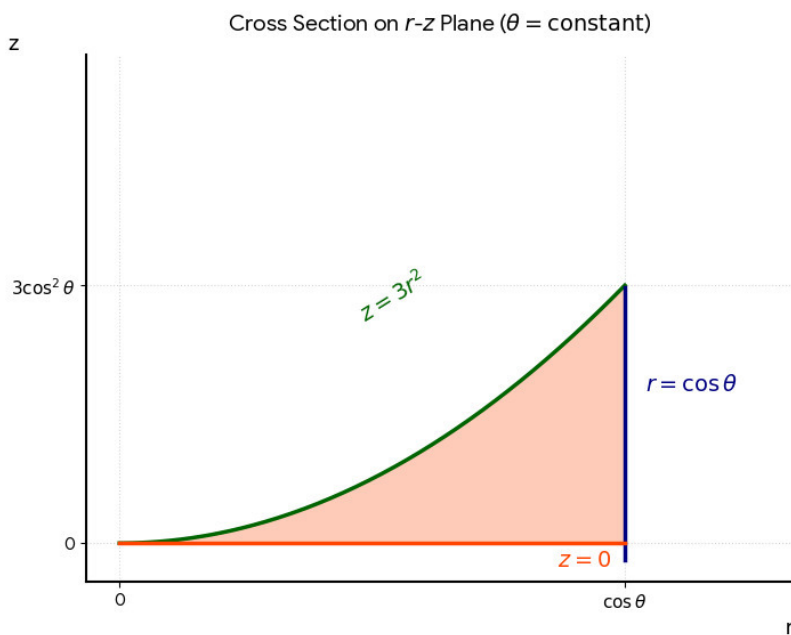
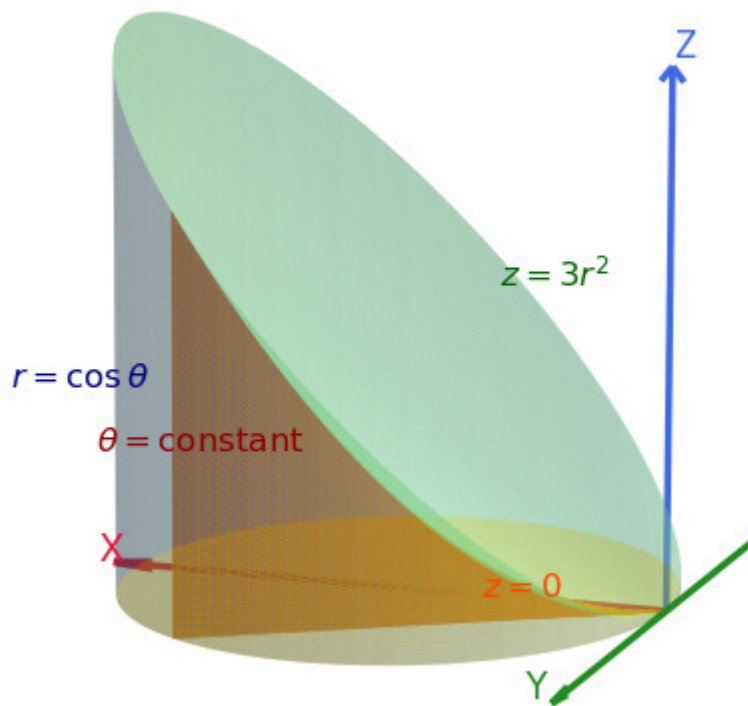
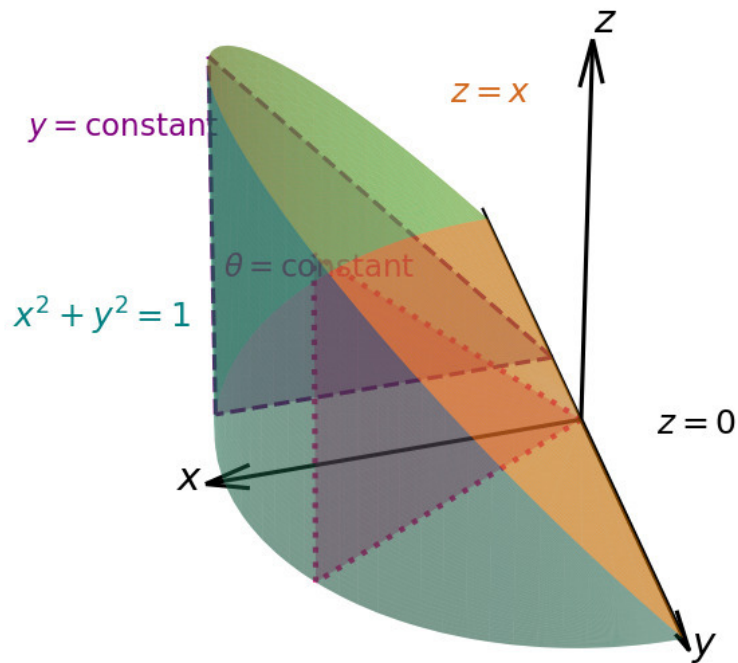
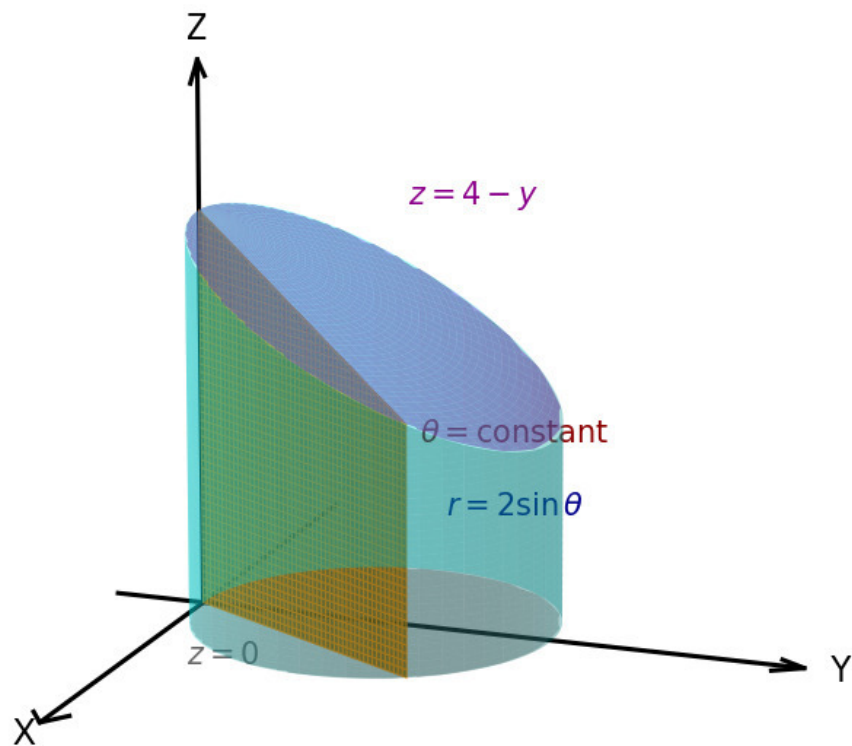


Figure 16: Solution to Section 15.7, problem 13. Answer = $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\cos \theta} \int_0^{3r^2} f(r, \theta, z) dz r dr d\theta$



$$\begin{aligned}
 & \int_{-1}^1 \int_0^{\sqrt{1-y^2}} \int_0^x (x^2 + y^2) dz dx dy \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_0^{\cos\theta} r^2 r dz dr d\theta \\
 &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 r^3 \cos\theta dr d\theta \\
 &= \frac{1}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta \\
 &= \frac{1}{2} \times \frac{4}{5} = \frac{2}{5} \#
 \end{aligned}$$

Figure 17: Solution to Section 15.7, problem 14. Construct the 3D image from $y = \text{constant}$ cross sections. Then construct $\theta = \text{constant}$ cross sections and find limits of integration.



Handwritten solution for problem 15 on a chalkboard. The domain D is defined as:

$$D = \begin{cases} \theta \in [0, \pi] \\ r \in [0, 2\sin\theta] \\ z \in [0, 4 - 2\sin^2\theta] \end{cases}$$

The volume element is $f \, dz \, r \, dr \, d\theta$. A diagram shows a circle in the yz -plane with the equation $y = r\sin\theta$.

Figure 18: Solution to Section 15.7, problem 15

Handwritten solution for problem 19 on a chalkboard. The volume is bounded by $0 \leq z \leq 2 - y = 2 - r\sin\theta$. A diagram shows a line $x = y$ in the xy -plane, with the angle θ between the x -axis and the line. The domain D is defined as:

$$D = \int_0^{\frac{\pi}{4}} \int_0^{\sec\theta} \int_0^{2 - r\sin\theta} r \, dz \, dr \, d\theta$$

The diagram also shows the polar coordinates $x = r\cos\theta$ and the condition $0 \leq x \leq 1 \Rightarrow 0 \leq r \leq \sec\theta$.

Figure 19: Solution to Section 15.7, problem 19