

Brief solutions to selected problems in homework 10

1. Section 14.10: Solutions, common mistakes and corrections:

15.2.3 $U = f(P, V, T)$
 $PV = nRT$
 (a) $\left(\frac{\partial U}{\partial P}\right)_V : U = f(P, V, T(P, V))$
 $\left(\frac{\partial U}{\partial P}\right)_V = f_P + f_T \cdot \left(\frac{\partial T}{\partial P}\right)_V$
 $= f_P + f_T \cdot \frac{V}{nR}$
 (b) $\left(\frac{\partial U}{\partial T}\right)_V : U = f(P(T, V), V, T)$
 $\left(\frac{\partial U}{\partial T}\right)_V = f_P \left(\frac{\partial P}{\partial T}\right)_V + f_T$
 $= f_P \cdot \frac{nR}{V} + f_T$

Figure 1: Solution to Section 14.10, problem 3

$x = r \cos \theta$
 $\left(\frac{\partial x}{\partial r}\right)_\theta = \cos \theta$
 $x^2 + y^2 = r^2$
 $r = \sqrt{x^2 + y^2}$
 $\left(\frac{\partial r}{\partial x}\right)_y = \frac{1}{2\sqrt{x^2 + y^2}} (2x)$
 $= \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}$

Figure 2: Solution to Section 14.10, problem 7

15.2.9 $f(x, y, z) = 0$

$(\frac{\partial x}{\partial y})_z: 0 = f(x(y, z), y, z)$
 $\Rightarrow 0 = f_x (\frac{\partial x}{\partial y})_z + f_y$
 $\Rightarrow (\frac{\partial x}{\partial y})_z = -\frac{f_y}{f_x}$

$(\frac{\partial y}{\partial z})_x: 0 = f(x, y(x, z), z)$
 $\Rightarrow 0 = f_y (\frac{\partial y}{\partial z})_x + f_z$
 $\Rightarrow (\frac{\partial y}{\partial z})_x = -\frac{f_z}{f_y}$

$(\frac{\partial z}{\partial x})_y: 0 = f(x, y, z(x, y))$
 $\Rightarrow 0 = f_x + f_z (\frac{\partial z}{\partial x})_y$
 $\Rightarrow (\frac{\partial z}{\partial x})_y = -\frac{f_x}{f_z}$

$\Rightarrow (\frac{\partial x}{\partial y})_z \cdot (\frac{\partial y}{\partial z})_x \cdot (\frac{\partial z}{\partial x})_y = -\frac{f_y}{f_x} \cdot \frac{f_z}{f_y} \cdot \frac{f_x}{f_z} = -1$

Figure 3: Solution to Section 14.10, problem 9

14.10

implies
 suppose x, y are independent

12 $f(x, y, z, w) = 0, \frac{\partial f}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = 0$
 $\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = 0$

$g(x, y, z, w) = 0, \frac{\partial g}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = 0$
 $\Rightarrow \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = 0$

$\begin{cases} \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = -\frac{\partial g}{\partial x} \end{cases} \Rightarrow \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial g}{\partial x} \end{pmatrix}$

$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

$\frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = -\frac{\partial f}{\partial x}$
 $\frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = -\frac{\partial g}{\partial x}$

$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix} \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial g}{\partial x} \end{pmatrix}$

$\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}^{-1} \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial g}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix}$

Figure 4: Solution to Section 14.10, problem 12

Remark: In other words, start with $f(x, y, z(x, y), w(x, y)) = 0, g(x, y, z(x, y), w(x, y)) = 0$, then proceed with ∂_x and ∂_y .

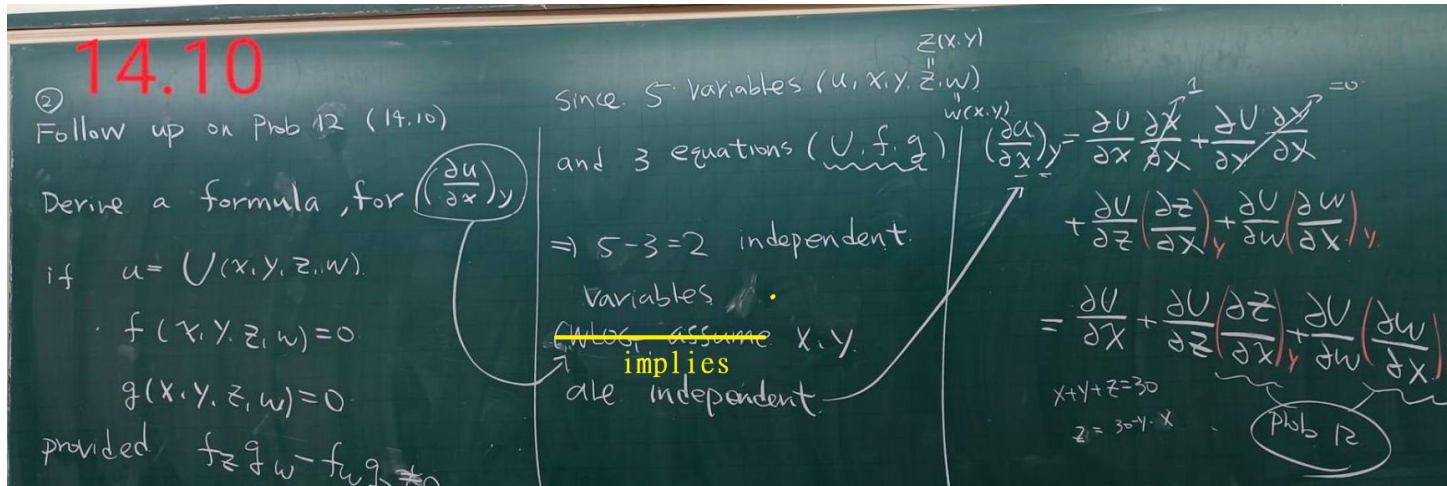


Figure 5: Solution to homework 10, problem 2

Remark: Start with $u = U(x, y, z(x, y), w(x, y)) = 0$, proceed with ∂_x , then substitute $\left(\frac{\partial w}{\partial x}\right)_y$ and $\left(\frac{\partial z}{\partial x}\right)_y$ computed from problem 12.

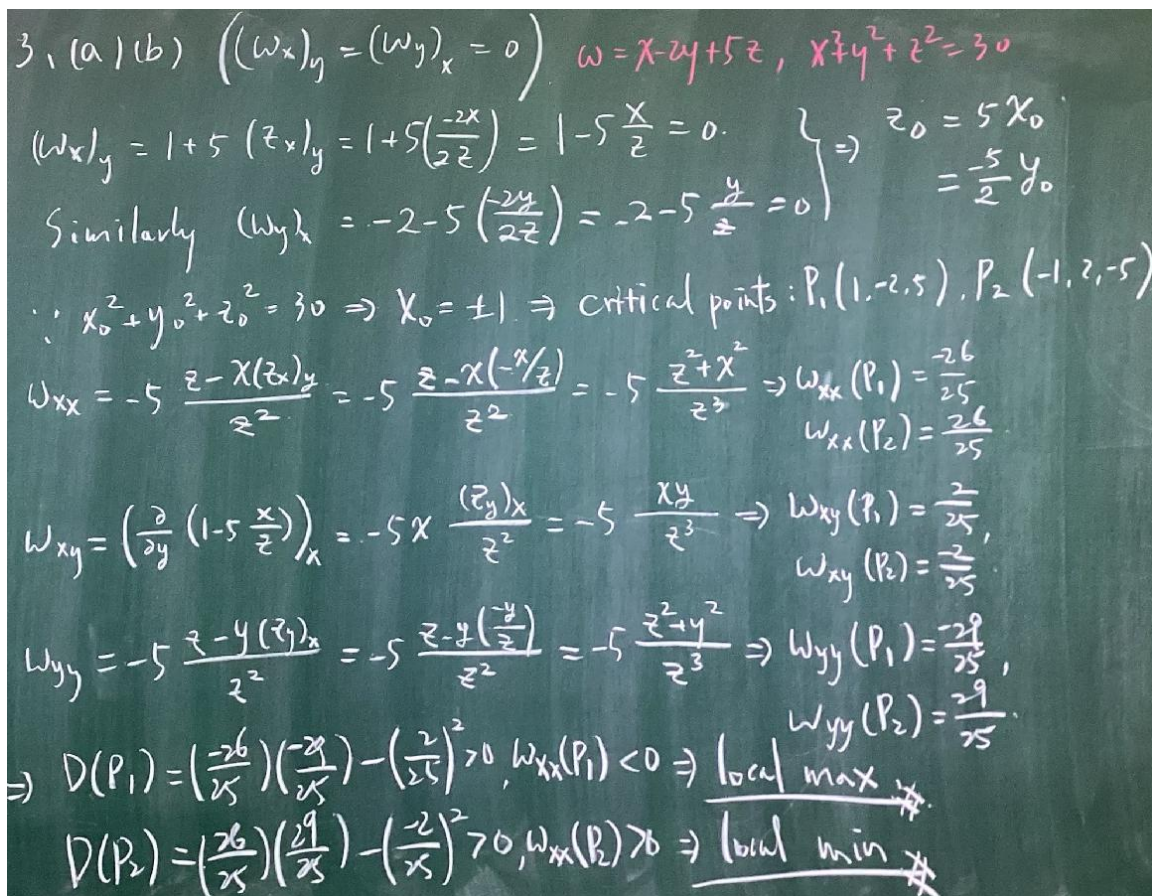


Figure 6: Solution to homework 10, problem 3

2. Section 15.1: Solutions, common mistakes and corrections:

$$\begin{aligned}
 & \int_{y=0}^2 \int_{x=0}^1 \frac{x}{1+xy} dx dy \\
 &= \int_{x=0}^1 \int_{y=0}^2 \frac{x}{1+xy} dy dx \\
 &= \int_0^1 \left[\ln(1+xy) \Big|_0^2 \right] dx \\
 &= \int_0^1 \ln(1+2x) dx \\
 & \underline{\underline{\hat{z} u=1+2x}} \int_{u=1}^3 \ln u \left(\frac{1}{2} du \right) \\
 &= \frac{1}{2} (u \ln u - u) \Big|_1^3 = \frac{3}{2} \ln 3 - 1
 \end{aligned}$$

Figure 7: Solution to Section 15.1, problem 33

$$\begin{aligned}
 F_x &= \frac{\partial}{\partial x} \left(\int_a^x \left(\int_c^y f(u,v) dv \right) du \right) \\
 &= \int_c^y f(x,v) dv \\
 F_y &= \frac{\partial}{\partial y} \left(\int_c^y \left(\int_a^x f(u,v) du \right) dv \right) = \int_a^x f(u,y) du \\
 \Rightarrow F_{xy} &= \frac{\partial}{\partial y} \left(\int_c^y f(x,v) dv \right) = f(x,y) \\
 F_{yx} &= \frac{\partial}{\partial x} \left(\int_a^x f(u,y) du \right) = f(x,y)
 \end{aligned}$$

Figure 8: Solution to Section 15.1, problem 36

3. Section 15.2: Solutions, common mistakes and corrections:

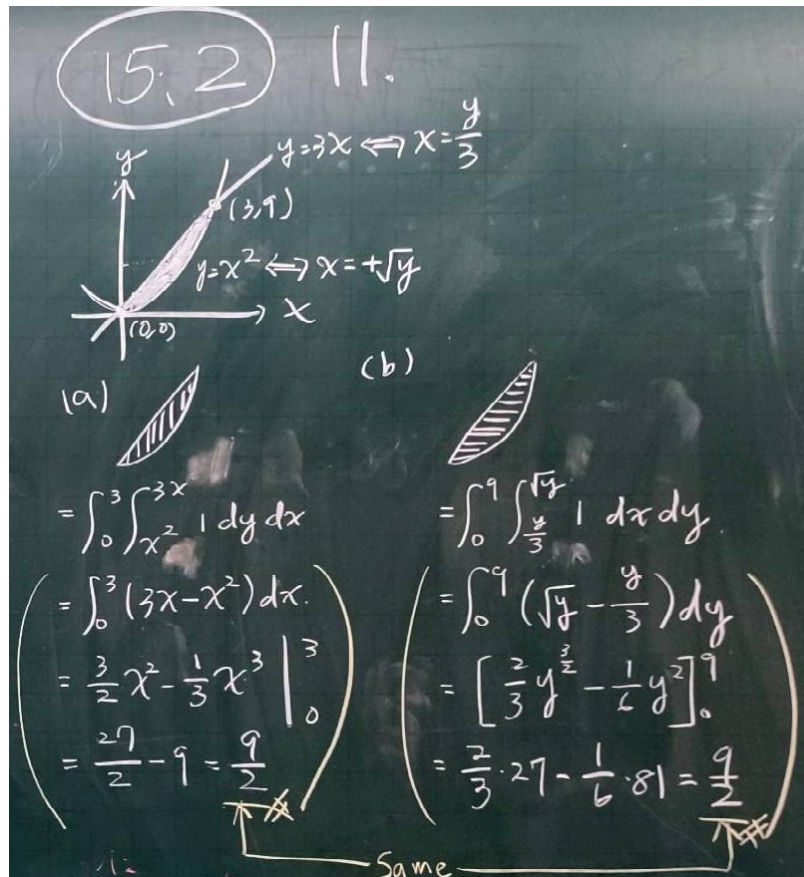


Figure 9: Solution to Section 15.2, problem 11

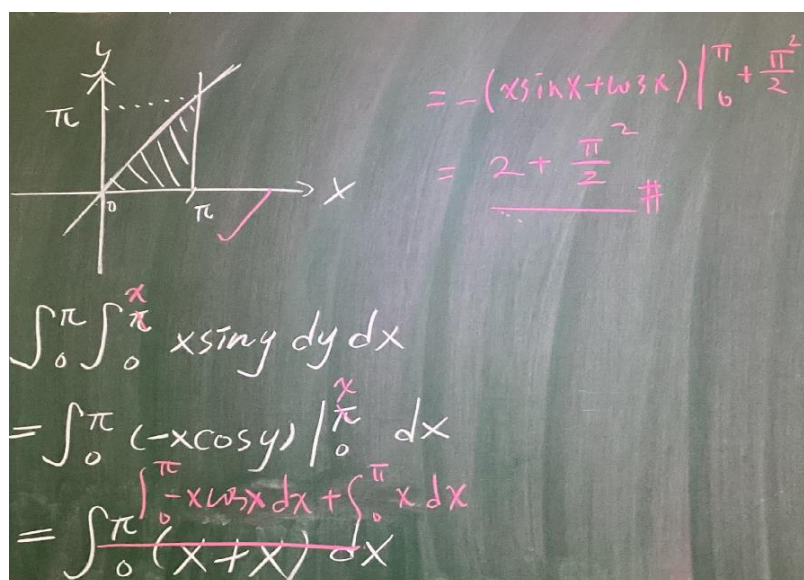


Figure 10: Solution to Section 15.2, problem 19

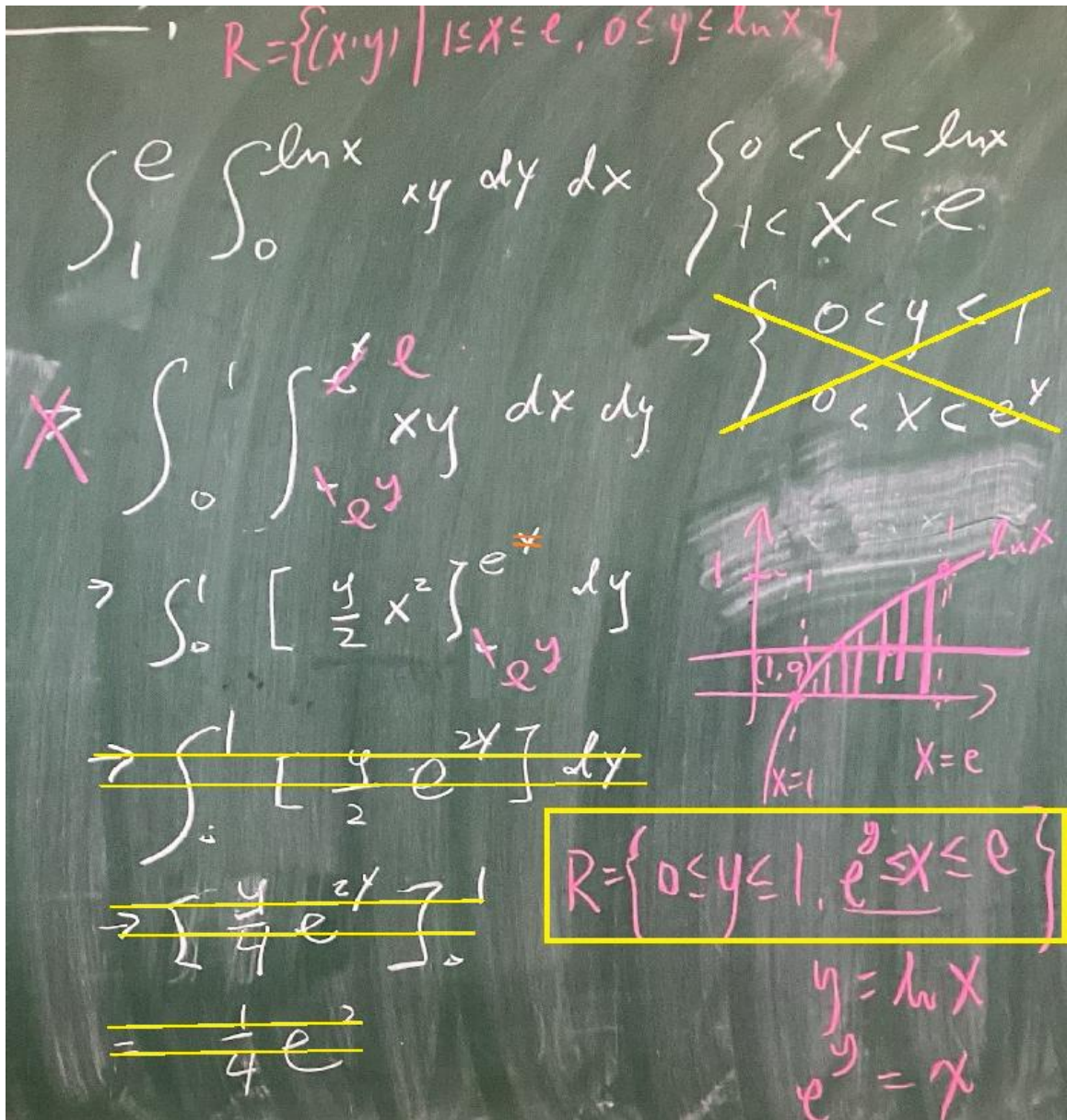


Figure 11: Solution to Section 15.2, problem 43

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx$$

$$0 \leq x \leq y \leq \pi$$

$$\int_0^\pi \int_0^y \frac{\sin y}{y} dx dy$$

$$= \int_0^\pi \frac{\sin y}{y} (y) dy$$

$$= \int_0^\pi \sin y dy$$

$$= [-\cos y]_0^\pi$$

$$= 2 \quad \checkmark$$

Figure 12: Solution to Section 15.2, problem 47

$$\int_1^\infty \int_1^x \frac{1}{x^2 y} dy dx = \int_1^\infty \frac{dx}{x^2} \int_1^x \frac{dy}{y}$$

$$= \int_1^\infty \frac{dx}{x^2} (\ln y) \Big|_1^x = \int_1^\infty \frac{1}{x^2} (0 - (-x)) dx$$

$$= \int_1^\infty \frac{dx}{x^2} = \lim_{b \rightarrow \infty} \int_1^b \frac{dx}{x^2} = \lim_{b \rightarrow \infty} \left. -\frac{1}{x} \right|_1^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + 1 \right)$$

Figure 13: Solution to Section 15.2, problem 69